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din Brașov

# HABILITATION THESIS SUMMARY

**Title:**

**Contributions to Complex Finsler Geometry. Models for Optimal Navigation under Gravity - A Finsler Approach**

**Domain: Mathematics**

**Author: Assoc. Prof. Dr. Codruţa Nicoleta Aldea**
**University: Transilvania University of Braşov**

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# Summary


The objective of this thesis is to present the author's main contributions to the development of complex Finsler geometry and to the extensions of Matsumoto's slope-of-a-mountain problem through a general model of time-optimal navigation based on Riemann-Finsler geometry, thereby establishing direct links with Zermelo's navigation problem.

In order to address some aspects related to complex Landsberg spaces, projectivity, holomorphic curvature, deformation, etc., different techniques from real Finsler approaches are applied, combined with the specific tools of complex Finsler geometry. The classic problems, Matsumoto's slope-of-a-mountain problem (MAT) and Zermelo's navigation problem (ZNP), presented independently in the literature, have been intensively explored within the framework of Riemann-Finsler geometry. The key argument is that in Finsler geometry, the notion of arc length can be interpreted as time, thus making the time-optimal paths locally the Finsler geodesics. The modern trend toward applications requires the development of new models. The main features of the navigation models described here are the type and range of compensation of the gravity effects on a mountain slope, which facilitate the description of various navigation problems and, in particular, link MAT and ZNP under the influence of gravity.

The thesis is divided into two parts: Part I. *Different aspects of complex Finsler geometry* which includes the first five chapters (Chapters 1-5) and Part II. *Extensions of Matsumoto's slope-of-a-mountain problem* encompassing four chapters (Chapters 6-9). At the end, a distinct chapter outlines some future research directions based on the topics discussed in the preceding chapters. Below, a brief description of each part of the thesis is presented.

**Part I.** This part comprises a few problems that we have studied in complex Finsler geometry, drawing heavily on our published papers [24, 25, 26, 27, 36, 23, 9, 14]. Chapter 1 briefly presents the main tools specific to complex Finsler geometry that are utilized throughout this section. In Chapter 2, we discuss complex Landsberg and generalized Berwald spaces, including particular instances of complex Landsberg spaces. Notable differences arise when compared to real reasoning, primarily due to the presence of two distinct horizontal covariant derivatives in complex Finsler geometry, specifically for Cartan tensors, one has $C_{i\bar{j}k|h}$ and $C_{i\bar{j}k|\bar{h}}$ with respect to Chern-Finsler connection. It is worthwhile to mention that the condition $C_{l\bar{r}h|k} = 0$ is equivalent to $C_{l\bar{r}h|\bar{k}} = 0$ and moreover, the horizontal coefficients $L^i_{jk}$ of the Chern-Finsler connection depend solely on the position coordinate $z$, in this case. This observation likely led T. Aikou to designate the complex Finsler spaces with $L^i_{jk} = L^i_{jk}(z)$ as complex Berwald spaces [7]. However, the defining characteristic of a complex Berwald space is that the horizontal coefficients $G^i_{jk}$ of a complex linear connection of Berwald type $B\Gamma$ are independent on the fiber coordinates, only within the Kähler context when $L^i_{jk} = G^i_{jk}$. Consequently, an unquestionable extension of complex Berwald spaces, directly linked to $B\Gamma$, is represented by a generalized Berwald space, characterized by $G^i_{jk}$ being dependent only on the position z.

To manage complex Landsberg spaces, another complex linear connection of Rund type $R\Gamma$ is utilized alongside $B\Gamma$, both tied to the canonical complex nonlinear connection. More precisely, a complex Landsberg space maintains the relationship $\overset{c}{L}{}^i_{jk} = G^i_{jk}$, which pertains to the horizontal coefficients of connections $R\Gamma$ and $B\Gamma$. To date, Kähler and Kähler-Berwald metrics are necessarily complex Landsberg metrics, yet the existence of a complex Landsberg metric (non-pure Hermitian), which is neither Kähler-Berwald nor Kähler, remains an unresolved issue. The general theory concerning generalized Berwald spaces is complemented by some special outcomes for the complex Randers metrics in Section 2.3. The results in this chapter are contained in the papers [26, 27].

The discussion on projectively related complex Finsler metrics begins in Chapter 3. Section 3.2 primarily delves into the complex variants of Rapcsák's theorem and develops a complex Finsler solution for Hilbert's fourth problem. Section 3.3 examines the projectivities of complex Randers metrics, $\tilde{F} = \alpha + |\beta|$, presenting the necessary and sufficient conditions for the metrics $\tilde{F}$ and $\alpha$ to be projectively related [25].

A more detailed analysis of the projective change relationship of complex Finsler metrics in Chapter 4 allows for the establishment of the existence of projective curvature invariants of Douglas and Weyl types. There are some formal similarities with studies from real Finsler geometry, but the differences between the real and complex cases are much more profound. More precisely, in Section 4.2, exploring the projective change relationship leads to three projective curvature invariants of Douglas type, and the vanishing of these characterizes complex Douglas spaces. This also allows for the derivation of additional properties for Kähler-Berwald spaces. Through a projective curvature invariant of Weyl type, a classification of Kähler-Berwald spaces of constant holomorphic curvature is achieved, whereby these spaces are either pure Hermitian if they have a non-null constant holomorphic curvature or non-pure Hermitian if they have null holomorphic curvature. Section 4.3 is dedicated to locally projectively flat complex Finsler metrics. In Section 4.4, an essential detail is the possibility of rewriting the equations of geodesic curves in a form that simplifies the study of complex Douglas spaces to the investigation of certain functions that arise from these equations. In Section 4.5, the general theory of complex Douglas spaces is applied to complex Randers spaces [24, 23].

In Chapter 5, we consider a problem of Zermelo navigation on a Hermitian manifold $(M, h)$, and we show that the solutions are real homogeneous functions, namely $\mathbb{R}$-complex Finsler metrics of Randers type (Section 5.3). Beyond the significance of the fact that Zermelo navigation provides a concrete application for the $\mathbb{R}$-complex Randers metrics, much more important is the fact that through it, non-pure Hermitian metrics (named $W$-Zermelo deformations) can be explicitly constructed. These are obtained by deforming the pure Hermitian metric $h$ through a vector field $W$. Section 5.4 presents this aspect, alongside the study of the invariance of certain properties of the pure Hermitian metrics as a result of $W$-deformations, considering particular vector fields $W$ [9, 14].

**Part II.** This part, based on the results obtained in our works [10, 20, 11, 12, 13], presents a collection of navigation problems on a slippery mountain slope represented by a Riemannian manifold $(M, h)$ of arbitrary dimension (at least 2), under the influence of "active winds", expressed through the gravitational wind (a gradient vector field) along with two traction coefficients. Chapter 6 outlines several basic notions and results from Riemann-Finsler geometry, which are necessary for the presentation of the subsequent chapters.

Before presenting Chapters 7-9, a brief description of the types of time-optimal navigation problems studied in the literature through Riemann-Finsler geometry is necessary, considering a particular case, specifically, in the presence of a gravitational wind. The concept of gravitational wind, recently introduced in the work [10], in the context of Zermelo navigation data [127, 45, 61], allows a unified description of all the time-optimal navigation problems presented in Chapters 7-9, including the classical ones (MAT and ZNP). The key aspect in describing the navigation models is the type and degree of compensation of the gravity effect on the mountain slope, which then characterizes the motion equations and, consequently, the corresponding Finsler metric for each case. We refer next to the two classical problems initially investigated by E. Zermelo and M. Matsumoto [156, 157, 106].

ZNP refers to the determination of the time-minimizing paths of a craft moving at a maximum speed relative to a surrounding and flowing medium, between two positions at sea, on the river or in the air, in the presence of a variable current (wind), modelled as a perturbing vector field $W$. The problem has been reformulated and generalized to Riemannian manifolds $(M, h)$ of arbitrary dimension, with solutions in Finsler geometry and spacetime [127, 71, 45, 87, 61, 124]. A gradient vector field can be treated as a special type of wind in the navigation data (h,W) [20]. This aligns with the concept of gravitational wind, which is the component $\mathbf{G}^T$ of the gravitational field. Thus, the general equation of motion is given by $v_{ZNP} = u + \mathbf{G}^T$, where $u$ denotes a self-velocity and $||u||_h = 1$ represents the maximum self-speed of a sailing or flying craft. The solution is provided by a Randers metric, whose indicatrix is the $h$-circle rigidly translated by $\mathbf{G}^T$.

MAT is also a time-minimization problem, where the objective is to determine the fastest paths on a slope of a mountain under the effect of gravity, taking into account that ascending is more exhausting than descending [106]. In this model, the transverse (lateral) component of the gravitational wind $\mathbf{G}^T$ (the cross-gravity additive) i.e. $\mathrm{Proj}_{u^\perp}\mathbf{G}^T$ is always cancelled and, therefore, has not impact on the resultant path, where $u^\perp$ is the direction orthogonal to the walker's self-velocity $u$. At the same time, the longitudinal component of $\mathbf{G}^T$ (the along-gravity effect) i.e. $\mathrm{Proj}_u\mathbf{G}^T$ (making evident that $\mathbf{G}^T = \mathrm{Proj}_u\mathbf{G}^T + \mathrm{Proj}_{u^\perp}\mathbf{G}^T$) is considered to act at full strength in any direction $u$ of motion, regardless of the wind force $||\mathbf{G}^T||_h$. This leads to the equation of motion $v_{MAT} = u + \mathrm{Proj}_u\mathbf{G}^T$, impling that the velocities $u$ and $v_{MAT}$ are always collinear, which contrasts with all other navigation problems described in this part. The solution is provided by the Matsumoto metric whose indicatrix is a limaçon in a two-dimensional model of the slope.

A direct connection between MAT and ZNP under the influence of a gravitational wind is presented in Chapter 7. Both problems are generalized and studied through a slippery slope model that incorporates a cross-traction coefficient, expressed by a real parameter $\eta \in [0, 1]$. In this model (slippery slope), the longitudinal component of the gravitational wind acts continuously, at full strength, in any direction of motion, regardless of the wind force $||\mathbf{G}^T||_h$, whereas the lateral component is subject to compensation due to traction, described by $\eta$. In this case, the equation of motion is given by $v_\eta = u + (1-\eta)\mathrm{Proj}_{u^\perp}\mathbf{G}^T + \mathrm{Proj}_u\mathbf{G}^T$, and the solution to the problem is provided by a Finsler metric called the slippery slope metric, which belongs to the class of general $(\alpha, \beta)$-metrics [10].

In Chapter 8, additional models for time-optimal navigation on a mountain slope are presented. First, a model is considered in which, unlike MAT, the transverse component of the gravitational wind is fully taken into account in the equation of motion, while the along-gravity effect is reduced completely. In this model, influenced solely by cross-gravity impact, referred to as cross slope (CROSS), the resultant velocity is given by $v_\dagger = u + \mathrm{Proj}_{u^\perp}\mathbf{G}^T$, and the solution to the problem is again provided by a general $(\alpha, \beta)$-metric, called the cross-slope metric [11]. Next, the fact that each of the two components of $\mathbf{G}^T$ can be partially reduced by introducing a traction coefficient is leveraged, rather than considering them entirely as in MAT (where only the lateral component is taken into account) or in CROSS (where only the longitudinal component is considered). Thus, by analogy with the slippery slope model from Chapter 7, another model, referred to as slippery cross slope, is explored in Section 8.3, concerning the along-gravity scaling by introducing an along-traction coefficient $\tilde{\eta} \in [0, 1]$. The equation of motion now becomes $v_{\tilde{\eta}} = u + \mathrm{Proj}_{u^\perp}\mathbf{G}^T + (1-\tilde{\eta})\mathrm{Proj}_u\mathbf{G}^T$ and the influence of the two components of the gravitational wind is somewhat reversed compared to the slippery slope model. Moreover, the slippery cross slope problem (whose solution is given by the slippery slope cross metric) directly connects CROSS and ZNP under the influence of the gravitational wind [12].

Chapter 9 provides a much more general model of navigation on the slippery slope of the mountain, which unifies and extends all the navigation problems developed in Chapters 7 and 8. In this case, it is now allowed for both components of the gravitational wind, relative to any direction $u$ of motion, to vary simultaneously in full ranges (both traction coefficients $\eta$, $\tilde{\eta} \in [0, 1]$ are now included in the general equation of motion). This scenario reflects the impact of both types of traction on the slippery slope, giving a much broader meaning to the problem of time-optimal navigation on the mountain slope [13].

A common characteristic of all the navigation problems studied in Chapters 7-9 is that their optimal solutions are provided by complex Finsler metrics belonging to the class of general $(\alpha, \beta)$-metrics (the so-called $(\eta, \tilde{\eta})$-slope metrics). These are obtained through a direction-dependent deformation of the background Riemannian metric $h$, followed by a rigid translation along a direction collinear with the gravitational wind.