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Title:

Contributions to Complex Finsler Geometry. Models for Optimal Navigation under Gravity - A Finsler Approach

Domain: Mathematics

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To the memory of my mother

Contents

(A) Rezumat

Obiectivul acestei teze este prezentarea principalelor contribuţii ale autorului la dezvoltarea geometriei Finsler complexe precum şi cele referitoare la extinderi ale problemei Matsumoto pe panta muntelui, printr-un model general de navigaţie optimă, bazat pe geometria Riemann-Finsler, stabilind legături directe cu problema de navigaţie Zermelo.

Pentru a studia probleme referitoare la spaţiile Landsberg complexe, proiectivitate, curbură olomorfă, deformări, etc., sunt combinate tehnici din geometria Finsler reală cu elemente specifice spaţiilor Finsler complexe. Problemele clasice (problema Matsumoto a pantei muntelui MAT şi problema de navigaţie Zermelo ZNP), prezentate in literatura de specialitate independent, sunt intens studiate prin intermediul geometriei Riemann-Finsler deoarece în geometria Finsler, noţiunea de lungime de arc desemnează timpul şi atunci, traiectoriile optime ca timp sunt local, geodezicele corespunzătoare metricilor Finsler. Tendinţele moderne spre aplicaţii, impun dezvoltatrea de noi modele. Caracteristicile principale ale modelelor de navigaţie descrise aici sunt tipul şi gradul de compensare ale efectului gravitaţiei asupra pantei muntelui. Acestea permit descrierea mai multor probleme de navigaţie şi în particular, leagă problema MAT de cea a navigaţiei Zermelo sub influenţa gravitaţiei.

Teza are două părţi: Partea I. *Câteva aspecte din geometria Finsler complexă* care cuprinde primele cinci capitole (Capitolele 1-5) şi Partea a II-a. *Extensii ale problemei Matsumoto a pantei muntelui* cu patru capitole (Capitolele 6-9). La finalul tezei, într-un capitol separat, sunt menţionate câteva direcţii de cercetare care au apărut pe parcursul descrierii rezultatelor din teză şi care ar putea fi dezvoltate. În continuare, prezentăm pe scurt fiecare parte a tezei.

Partea I. Această parte cuprinde câteva probleme pe care le-am studiat în geometria Finsler complexă, rezultatele descrise aici fiind publicate în articolele [24, 25, 26, 27, 36, 23, 9, 14]. Capitolul 1 prezintă pe scurt principalele instrumente, specifice geometriei Finsler complexe, care sunt utilizate de-a lungul acestei părţi. În Capitolul 2 prezentăm spaţiile Landsberg complexe şi Berwald generalizate, precum şi câteva cazuri particulare de spaţii Landsberg complexe. Între situaţiile care apar în cazul complex, comparativ cu cele din cazul real, există deosebiri semnificative, numai dacă se ia în seamă faptul că în geometria Finsler complexă există două derivate covariante orizontale diferite (conjugate), în particular pentru tensorii Cartan complecşi acestea sunt $C_{i\bar{j}k|h}$ şi $C_{i\bar{j}k|\bar{h}}$, în raport cu conexiunea Chern-Finsler. Este important de menţionat faptul că o condiţie de forma $C_{l\bar{r}h|k} = 0$ este echivalentă cu $C_{l\bar{r}h|\bar{k}} = 0$ şi mai mult, în acest caz, coeficienţii orizontali L^i_{jk} ai conexiunii Chern-Finsler depind doar de punctele z de pe varietate. Probabil că acesta a fost principalul motiv care l-a determinat pe T. Aikou să denumească spaţiile Finsler complexe care au proprietatea că $L^i_{jk} = L^i_{jk}(z)$, spaţii Berwald complexe [7]. Cu toate acestea, condiţia ca un spaţiu să fie Berwald complex poate fi exprimată prin independeţa coeficienţilor orizontali G^i_{jk} ai unei conexiuni liniare de tip Berwald $B\Gamma$, de coordonatele direcţiilor tangente, doar în cazul în care spaţiul este Kähler,

adică atunci când $L^i_{jk} = G^i_{jk}$. Aşadar, o extensie incontestabilă a spaţiilor Berwald complexe, în legătură directă cu $B\Gamma$, este dată de spaţiile Berwald generalizate caracterizate prin faptul că funcţiile G^i_{jk} depind doar de poziţia z. Pentru a studia spaţiile Landsberg complexe, utilizăm alături de $B\Gamma$, şi o altă conexiune liniară complexă de tip Rund $R\Gamma$, amândouă fiind asociate conexiunii neliniare canonice. Mai exact, un spaţiu Landsberg complex satisface condiţia $\overset{c}{L^i_{jk}} = G^i_{jk}$ care se referă la coeficienţii orizontali ai conexiunilor $R\Gamma$ şi $B\Gamma$. Până acum se cunoaşte faptul că metricile Kähler şi Kähler-Berwald sunt metrici Landsberg complexe, dar existenţa unor exemple de metrici Landsberg complexe care nu sunt nici Kähler-Berwald şi nici Kähler este o problemă deschisă. Teoria generală a spaţiilor Berwald generalizate este completată cu câteva rezultate speciale referitoare la meticile Randers complexe în Secţiunea 2.3. Rezultatele din acest capitol sunt publicate în articolele [26, 27].

Problema metricilor Finsler complexe proiectiv echivalente este prezentată începând cu Capitolul 3. Secţiunea 3.2 este preponderent canalizată pe versiunile complexe ale teoremei Rapcsák şi pe o soluţie Finsler complexă a problemei a patra a lui Hilbert. Secţiunea 3.3 explorează proiectivităţile metricilor Randers complexe $\tilde{F} = \alpha + |\beta|$, un rezultat fiind dat de condiţiile necesare şi suficiente ca metricile $\tilde{F}$ şi α să fie proiecctiv echivalente [25].

O analiză mai amănunţită a relaţiei de echivalenţă proiectivă a metricilor Finsler complexe permite stabilirea existenţei unor invarianţi proiectivi de curbură de tip Douglas şi de tip Weyl, în Capitolul 4. Există unele similitudini formale cu studiile din geometria Finsler reală, dar deosebirile dintre cazul real şi cel complex sunt mult mai profunde. Mai exact, în Secţiunea 4.2 explorarea relaţiei de echivalenţă proiectivă conduce la trei invariaţi proiectivi de curbură de tip Douglas, iar anularea acestora caracterizează spaţiile Douglas complexe. De asemenea, aceasta permite şi obţinerea unor proprietăţi suplimentare pentru spaţiile Kähler-Berwald. Prin intermediul unui invariant proiectiv de curbură de tip Weyl se obţine o clasificare a spaţiile Kähler-Berwald de curbură olomorfă constantă şi anume, acestea sunt pur hermitiene, dacă au curbura olomorfă o constantă nenulă sau non-pur hermitiene, dacă au curbura olomorfă nulă. Secţiunea 4.3 este dedicată metricilor Finsler complexe local proiectiv plate. În Secţiunea 4.4 un detaliu esenţial este posibilitatea scrierii ecuaţiilor curbelor geodezice sub o anumită formă care reduce studiul spaţiilor Douglas complexe, la investigarea unor funcţii care provin din aceste ecuaţii. În Secţiunea 4.5 teoria generală a spaţiilor Douglas complexe este aplicată spaţiilor Randers complexe [24, 23].

În Capitolul 5 considerăm o problemă de navigaţie Zermelo pe o varietate hermitiană (M, h) şi arătăm că soluţiile sunt funcţii real omogene, adică $\mathbb{R}$-metrici Finsler complexe de tip Randers (Secţiunea 5.3). Dincolo de semnificaţia faptului că navigaţia Zermelo furnizează o aplicaţie concretă a $\mathbb{R}$-metricilor Randers, mult mai important este faptul că prin intermediul acesteia pot fi construite explicit metrici non-hermitiene (numite W-deformări Zermelo), obţinute prin deformarea metricii hermitiene h, printr-un câmp vectorial W. În Secţiunea 5.4 este prezentat acest aspect, alături de studiul invarianţei unor proprietăţi ale metricilor hermitiene, ca urmare a W-deformărilor, considerând câmpuri vectoriale W particulare [9, 14].

Partea a II-a. Această parte, bazată pe rezultatele obţinute în lucrările [10, 20, 11, 12, 13], prezintă o colecţie de probleme de navigaţie pe panta alunecoasă (versantul alunecos) a unui munte, reprezentat de o varietate riemanniană (M, h) de dimensiune cel puţin doi, sub acţiunea unor "vânturi active", exprimate prin intermediul vântului gravitaţional (un câmp vectorial gradient), împreună cu doi coeficienţi de tracţiune. Capitolul 6 punctează câteva noţiuni şi rezultate de bază din geometria Riemann-Finsler, acestea fiind necesare în prezentarea celorlalte capitole.

Înainte de a prezenta Capitolele 7-9, se impune o scurtă descriere a tipurilor de probleme de navigaţie optimă în raport cu timpul, studiate în literatură prin intermediul geometriei Riemann-Finsler, considerând un caz particular, mai exact, în prezenţa unui vânt gravitaţional. Noţiunea de vânt gravitaţional introdusă recent în lucrarea [10], în contextul datelor de navigaţie Zermelo [127, 45, 71, 124, 61] permite o descriere unitară a tuturor problemelor de navigaţie optimă prezentate în Capitolele 7-9, incluzându-le totodată şi pe cele clasice (MAT şi ZNP). Aspectul cheie în descrierea modelelor de navigaţie este dat de tipul şi gradul de compensare ale efectului gravitaţiei asupra pantei muntelui care apoi, caracterizează ecuaţiile de mişcare şi în consecinţă, metrica Finsler corespunzătoare fiecărui caz. În continuare ne referim la cele două probleme clasice, investigate iniţial de E. Zermelo respectiv, M. Matsumoto [156, 157, 106].

ZNP se referă la determinarea celor mai rapide traiectorii ale unei ambarcaţiuni care se deplasează cu o viteză maximă în raport cu un mediu înconjurător, între două locaţii pe mare sau în aer, în prezenţa unui curent (vânt) variabil, exprimat printr-un câmp vectorial W. Problema a fost reformulată şi generalizată la varietăţi riemanniene (M, h) de dimensiune arbitrară, cu soluţii în geometria Finsler şi spaţiu-timp [127, 71, 45, 87, 61, 124]. Un câmp vectorial gradient poate fi tratat ca un vânt special în datele de navigaţie (h, W) [20]. Aceasta se pliează cu noţiunea de vânt gravitaţional care este componenta $\mathbf{G}^T$ a câmpului gravitaţional. Atunci, ecuaţia generală de mişcare este $v_{ZNP} = u + \mathbf{G}^T$, unde u reprezintă vectorul viteză proprie, cu viteza maximă $||u||_h = 1$. Soluţia este dată prin intermediul unei metrici Randers a cărei indicatoare este h-cercul translatat cu $\mathbf{G}^T$.

MAT este tot o problemă de minimizare în raport cu timpul, fiind urmărite cele mai rapide drumuri care pot fi parcurse pe un versant al unui munte, sub efectul gravitaţiei, ţinând cont de faptul că a urca este mai obositor decât a coborî [106]. În acest model, componenta transversală a vântului gravitaţional (cross-gravity additive), i.e. $\mathrm{Proj}_{u^\perp}\mathbf{G}^T$ (direcţia $u^\perp$ fiind ortogonală lui u) este întotdeauna anulată şi, prin urmare, nu are niciun impact asupra traseului rezultant. În acelaşi timp, componenta longitudinală a vântului gravitaţional (along-gravity effect), i.e. $\mathrm{Proj}_u\mathbf{G}^T$ (fiind evident că $\mathbf{G}^T = \mathrm{Proj}_u\mathbf{G}^T + \mathrm{Proj}_{u^\perp}\mathbf{G}^T$) este considerată maximă, în orice direcţie u de mişcare, oricare ar fi forţa vântului $||\mathbf{G}^T||_h$. Acest fapt conduce la ecuaţia de mişcare $v_{MAT} = u + \mathrm{Proj}_u\mathbf{G}^T$ şi de asemenea, implică faptul că vitezele u şi v_{MAT} sunt întotdeauna coliniare, ceea ce contrastează cu toate celelalte probleme de navigaţie descrise în această parte. Soluţia este dată prin intermediul metricii Matsumoto a cărei indicatoare este curba limaçon, în cazul unui model 2-dimensional.

O legătură directă între MAT şi ZNP sub influenţa vântului gravitaţional este prezentată în Capitolul 7. Amândouă problemele sunt generalizate şi studiate printr-un model al versantului alunecos care include un coeficient de tracţiune transversal (cross-traction) exprimat prin intermediul unui parametru real $\eta \in [0, 1]$. În aceast model (slippery slope), componenta longitudinală a vântului gravitaţional acţionează continuu, cu toată puterea în orice direcţie de mişcare, oricare ar fi forţa vântului $||\mathbf{G}^T||_h$, pe când, componenta laterală este supusă compensării, din cauza tracţiunii descrisă prin η. În acest caz, ecuaţia generală de mişcare este $v_\eta = u + (1-\eta)\mathrm{Proj}_{u^\perp}\mathbf{G}^T + \mathrm{Proj}_u\mathbf{G}^T$, iar soluţia problemei este dată de o metrica numită slippery slope, aceasta fiind o (α, β)-metrică generală [10].

În Capitolul 8 sunt prezentate şi alte modele de navigaţie optimă, pe panta unui munte. Mai întâi este considerat un model în care, spre deosebire de MAT, componenta transversală a vântului gravitaţional este luată în considerare în întregime în ecuaţia de mişcare, în timp ce componenta longitudinală este ignorată. În acest model, acţionat doar de $\mathrm{Proj}_{u^\perp}\mathbf{G}^T$ (cross-gravity), numit CROSS, viteza rezultantă este $v_\dagger = u + \mathrm{Proj}_{u^\perp}\mathbf{G}^T$, iar soluţia problemei este dată tot de o metrică (α, β)-generală, (metrica cross-slope) [11]. Apoi este valorificat faptul că fiecare dintre cele două componente ale lui $\mathbf{G}^T$ pot fi reduse parţial prin introducerea unui coeficient de tracţiune, nu doar luate în întregime ca în MAT (doar componenta laterală) sau în CROSS (doar componenta longitudinală). Aşadar, prin analogie cu modelul slippery slope din Chapter 7, un alt model denumit slippery cross slope este explorat în Secţiunea 8.3, introducând un alt coeficient de tracţiune $\tilde{\eta} \in [0, 1]$, numit along-traction. Ecuaţia de mişcare este acum $v_{\tilde{\eta}} = u + \mathrm{Proj}_{u^\perp}\mathbf{G}^T + (1 - \tilde{\eta})\mathrm{Proj}_u\mathbf{G}^T$, iar influenţa celor două componente ale vântului gravitaţional este aici orecum inversată, comparativ cu modelul slippery slope. Mai mult, problema slippery cross slope (soluţia acesteia fiind dată de metrica numită slippery slope cross) leagă direct CROSS şi ZNP sub influenţa vântului gravitational [12].

Capitolul 9 oferă un model mult mai general de navigaţie pe panta alunecoasă a muntelui, care uneşte şi extinde toate problemele de navigație dezvoltate în Capitolele 7 şi 8. Acum este admisă situaţia ca ambele componente ale vântului gravitaţional, în raport cu direcţia de mişcare, să varieze simultan pe intervale complete (ambii coeficienţi de tracțiune η, $\tilde{\eta} \in [0, 1]$ sunt acum incluşi în ecuaţia generală de mişcare). Acest scenariu reflectă impactul ambelor tracțiuni pe versantul alunecos, ceea ce conferă un sens mult mai larg problemei de navigaţie optimă, în raport cu timpul pe panta muntelui [13].

Caracteristica comună tuturor problemelor de navigaţie (studiate în Capitolele 7-9) este că soluţiile optime ale acestora sunt furnizate de metrici Finsler complexe din clasa (α, β)-metricilor generale (aşa-numitele $(\eta, \tilde{\eta})$-slope metrics). Acestea sunt obţinute printr-o deformare a metricii riemanniene h, dependentă de direcţia de mişcare u, urmată apoi de o translaţie rigidă, dată de o direcţie coliniară vântului gravitaţional.

(A-i) Summary

The objective of this thesis is to present the author's main contributions to the development of complex Finsler geometry and to the extensions of Matsumoto's slope-of-a-mountain problem through a general model of time-optimal navigation based on Riemann-Finsler geometry, thereby establishing direct links with Zermelo's navigation problem.

In order to address some aspects related to complex Landsberg spaces, projectivity, holomorphic curvature, deformation, etc., different techniques from real Finsler approaches are applied, combined with the specific tools of complex Finsler geometry. The classic problems, Matsumoto's slope-of-a-mountain problem (MAT) and Zermelo's navigation problem (ZNP), presented independently in the literature, have been intensively explored within the framework of Riemann-Finsler geometry. The key argument is that in Finsler geometry, the notion of arc length can be interpreted as time, thus making the time-optimal paths locally the Finsler geodesics. The modern trend toward applications requires the development of new models. The main features of the navigation models described here are the type and range of compensation of the gravity effects on a mountain slope, which facilitate the description of various navigation problems and, in particular, link MAT and ZNP under the influence of gravity.

The thesis is divided into two parts: Part I. *Different aspects of complex Finsler geometry* which includes the first five chapters (Chapters 1-5) and Part II. *Extensions of Matsumoto's slope-of-a-mountain problem* encompassing four chapters (Chapters 6-9). At the end, a distinct chapter outlines some future research directions based on the topics discussed in the preceding chapters. Below, a brief description of each part of the thesis is presented.

Part I. This part comprises a few problems that we have studied in complex Finsler geometry, drawing heavily on our published papers [24, 25, 26, 27, 36, 23, 9, 14]. Chapter 1 briefly presents the main tools specific to complex Finsler geometry that are utilized throughout this section. In Chapter 2, we discuss complex Landsberg and generalized Berwald spaces, including particular instances of complex Landsberg spaces. Notable differences arise when compared to real reasoning, primarily due to the presence of two distinct horizontal covariant derivatives in complex Finsler geometry, specifically for Cartan tensors, one has $C_{i\bar{j}k|h}$ and $C_{i\bar{j}k|\bar{h}}$ with respect to Chern-Finsler connection. It is worthwhile to mention that the condition $C_{l\bar{r}h|k} = 0$ is equivalent to $C_{l\bar{r}h|\bar{k}} = 0$ and moreover, the horizontal coefficients L^i_{jk} of the Chern-Finsler connection depend solely on the position coordinate z, in this case. This observation likely led T. Aikou to designate the complex Finsler spaces with $L^i_{jk} = L^i_{jk}(z)$ as complex Berwald spaces [7]. However, the defining characteristic of a complex Berwald space is that the horizontal coefficients G^i_{jk} of a complex linear connection of Berwald type $B\Gamma$ are independent on the fiber coordinates, only within the Kähler context when $L^i_{jk} = G^i_{jk}$. Consequently, an unquestionable extension of complex Berwald spaces, directly linked to $B\Gamma$, is represented by a generalized Berwald space, characterized by G^i_{jk} being dependent only on the position z.

To manage complex Landsberg spaces, another complex linear connection of Rund type $R\Gamma$ is utilized alongside $B\Gamma$, both tied to the canonical complex nonlinear connection. More precisely, a complex Landsberg space maintains the relationship $\overset{c}{L}{}^i_{jk} = G^i_{jk}$, which pertains to the horizontal coefficients of connections $R\Gamma$ and $B\Gamma$. To date, Kähler and Kähler-Berwald metrics are necessarily complex Landsberg metrics, yet the existence of a complex Landsberg metric (non-pure Hermitian), which is neither Kähler-Berwald nor Kähler, remains an unresolved issue. The general theory concerning generalized Berwald spaces is complemented by some special outcomes for the complex Randers metrics in Section 2.3. The results in this chapter are contained in the papers [26, 27].

The discussion on projectively related complex Finsler metrics begins in Chapter 3. Section 3.2 primarily delves into the complex variants of Rapcsák's theorem and develops a complex Finsler solution for Hilbert's fourth problem. Section 3.3 examines the projectivities of complex Randers metrics, $\tilde{F} = \alpha + |\beta|$, presenting the necessary and sufficient conditions for the metrics $\tilde{F}$ and α to be projectively related [25].

A more detailed analysis of the projective change relationship of complex Finsler metrics in Chapter 4 allows for the establishment of the existence of projective curvature invariants of Douglas and Weyl types. There are some formal similarities with studies from real Finsler geometry, but the differences between the real and complex cases are much more profound. More precisely, in Section 4.2, exploring the projective change relationship leads to three projective curvature invariants of Douglas type, and the vanishing of these characterizes complex Douglas spaces. This also allows for the derivation of additional properties for Kähler-Berwald spaces. Through a projective curvature invariant of Weyl type, a classification of Kähler-Berwald spaces of constant holomorphic curvature is achieved, whereby these spaces are either pure Hermitian if they have a non-null constant holomorphic curvature or non-pure Hermitian if they have null holomorphic curvature. Section 4.3 is dedicated to locally projectively flat complex Finsler metrics. In Section 4.4, an essential detail is the possibility of rewriting the equations of geodesic curves in a form that simplifies the study of complex Douglas spaces to the investigation of certain functions that arise from these equations. In Section 4.5, the general theory of complex Douglas spaces is applied to complex Randers spaces [24, 23].

In Chapter 5, we consider a problem of Zermelo navigation on a Hermitian manifold (M, h), and we show that the solutions are real homogeneous functions, namely $\mathbb{R}$-complex Finsler metrics of Randers type (Section 5.3). Beyond the significance of the fact that Zermelo navigation provides a concrete application for the $\mathbb{R}$-complex Randers metrics, much more important is the fact that through it, non-pure Hermitian metrics (named W-Zermelo deformations) can be explicitly constructed. These are obtained by deforming the pure Hermitian metric h through a vector field W. Section 5.4 presents this aspect, alongside the study of the invariance of certain properties of the pure Hermitian metrics as a result of W-deformations, considering particular vector fields W [9, 14].

Part II. This part, based on the results obtained in our works [10, 20, 11, 12, 13], presents a collection of navigation problems on a slippery mountain slope represented by a Riemannian manifold (M, h) of arbitrary dimension (at least 2), under the influence of "active winds", expressed through the gravitational wind (a gradient vector field) along with two traction coefficients. Chapter 6 outlines several basic notions and results from Riemann-Finsler geometry, which are necessary for the presentation of the subsequent chapters.

Before presenting Chapters 7-9, a brief description of the types of time-optimal navigation problems studied in the literature through Riemann-Finsler geometry is necessary, considering a particular case, specifically, in the presence of a gravitational wind. The concept of gravitational wind, recently introduced in the work [10], in the context of Zermelo navigation data [127, 45, 61], allows a unified description of all the time-optimal navigation problems presented in Chapters 7-9, including the classical ones (MAT and ZNP). The key aspect in describing the navigation models is the type and degree of compensation of the gravity effect on the mountain slope, which then characterizes the motion equations and, consequently, the corresponding Finsler metric for each case. We refer next to the two classical problems initially investigated by E. Zermelo and M. Matsumoto [156, 157, 106].

ZNP refers to the determination of the time-minimizing paths of a craft moving at a maximum speed relative to a surrounding and flowing medium, between two positions at sea, on the river or in the air, in the presence of a variable current (wind), modelled as a perturbing vector field W. The problem has been reformulated and generalized to Riemannian manifolds (M, h) of arbitrary dimension, with solutions in Finsler geometry and spacetime [127, 71, 45, 87, 61, 124]. A gradient vector field can be treated as a special type of wind in the navigation data (h,W) [20]. This aligns with the concept of gravitational wind, which is the component $\mathbf{G}^T$ of the gravitational field. Thus, the general equation of motion is given by $v_{ZNP} = u + \mathbf{G}^T$, where u denotes a self-velocity and $||u||_h = 1$ represents the maximum self-speed of a sailing or flying craft. The solution is provided by a Randers metric, whose indicatrix is the h-circle rigidly translated by $\mathbf{G}^T$.

MAT is also a time-minimization problem, where the objective is to determine the fastest paths on a slope of a mountain under the effect of gravity, taking into account that ascending is more exhausting than descending [106]. In this model, the transverse (lateral) component of the gravitational wind $\mathbf{G}^T$ (the cross-gravity additive) i.e. $\mathrm{Proj}_{u^\perp}\mathbf{G}^T$ is always cancelled and, therefore, has not impact on the resultant path, where $u^\perp$ is the direction orthogonal to the walker's self-velocity u. At the same time, the longitudinal component of $\mathbf{G}^T$ (the along-gravity effect) i.e. $\mathrm{Proj}_u\mathbf{G}^T$ (making evident that $\mathbf{G}^T = \mathrm{Proj}_u\mathbf{G}^T + \mathrm{Proj}_{u^\perp}\mathbf{G}^T$) is considered to act at full strength in any direction u of motion, regardless of the wind force $||\mathbf{G}^T||_h$. This leads to the equation of motion $v_{MAT} = u + \mathrm{Proj}_u\mathbf{G}^T$, impling that the velocities u and v_{MAT} are always collinear, which contrasts with all other navigation problems described in this part. The solution is provided by the Matsumoto metric whose indicatrix is a limaçon in a two-dimensional model of the slope.

A direct connection between MAT and ZNP under the influence of a gravitational wind is presented in Chapter 7. Both problems are generalized and studied through a slippery slope model that incorporates a cross-traction coefficient, expressed by a real parameter $\eta \in [0, 1]$. In this model (slippery slope), the longitudinal component of the gravitational wind acts continuously, at full strength, in any direction of motion, regardless of the wind force $||\mathbf{G}^T||_h$, whereas the lateral component is subject to compensation due to traction, described by η. In this case, the equation of motion is given by $v_\eta = u + (1 - \eta)\mathrm{Proj}_{u^\perp}\mathbf{G}^T + \mathrm{Proj}_u\mathbf{G}^T$, and the solution to the problem is provided by a Finsler metric called the slippery slope metric, which belongs to the class of general (α, β)-metrics [10].

In Chapter 8, additional models for time-optimal navigation on a mountain slope are presented. First, a model is considered in which, unlike MAT, the transverse component of the gravitational wind is fully taken into account in the equation of motion, while the along-gravity effect is reduced completely. In this model, influenced solely by cross-gravity impact, referred to as cross slope (CROSS), the resultant velocity is given by $v_\dagger = u + \mathrm{Proj}_{u^\perp}\mathbf{G}^T$, and the solution to the problem is again provided by a general (α, β)-metric, called the cross-slope metric [11]. Next, the fact that each of the two components of $\mathbf{G}^T$ can be partially reduced by introducing a traction coefficient is leveraged, rather than considering them entirely as in MAT (where only the lateral component is taken into account) or in CROSS (where only the longitudinal component is considered). Thus, by analogy with the slippery slope model from Chapter 7, another model, referred to as slippery cross slope, is explored in Section 8.3, concerning the along-gravity scaling by introducing an along-traction coefficient $\tilde{\eta} \in [0, 1]$. The equation of motion now becomes $v_{\tilde{\eta}} = u + \mathrm{Proj}_{u^{\perp}} \mathbf{G}^T + (1 - \tilde{\eta}) \mathrm{Proj}_u \mathbf{G}^T$ and the influence of the two components of the gravitational wind is somewhat reversed compared to the slippery slope model. Moreover, the slippery cross slope problem (whose solution is given by the slippery slope cross metric) directly connects CROSS and ZNP under the influence of the gravitational wind [12].

Chapter 9 provides a much more general model of navigation on the slippery slope of the mountain, which unifies and extends all the navigation problems developed in Chapters 7 and 8. In this case, it is now allowed for both components of the gravitational wind, relative to any direction u of motion, to vary simultaneously in full ranges (both traction coefficients η, $\tilde{\eta} \in [0, 1]$ are now included in the general equation of motion). This scenario reflects the impact of both types of traction on the slippery slope, giving a much broader meaning to the problem of time-optimal navigation on the mountain slope [13].

A common characteristic of all the navigation problems studied in Chapters 7-9 is that their optimal solutions are provided by complex Finsler metrics belonging to the class of general (α, β)-metrics (the so-called $(\eta, \tilde{\eta})$-slope metrics). These are obtained through a direction-dependent deformation of the background Riemannian metric h, followed by a rigid translation along a direction collinear with the gravitational wind.

(B) Scientific and professional achievements. The evolution and development plans for career development

(B-i) Scientific and professional achievements

Introduction

The scientific results included here represent a collection of outcomes of the author in the complex and real Finsler geometries. Th presentation is structured into two parts: Part I. *Different aspects of complex Finsler geometry* which comprises the first five chapters (Chapters 1-5) and Part II. *Extensions of Matsumoto's slope-of-a-mountain problem* with four chapters (Chapters 6-9), which aim to show separately the contributions of the author in both branches of Finsler geometry (complex and real). Although somewhat unusual, we start (in Part I) by presenting some results obtained in complex Finsler geometry - the principal background of the author. Motivated by the modern trend toward applications, in recent years the author has expanded her research area to also include the older and more widely known branch of Finsler geometry, namely the real Finsler geometry, proposing an original idea that generalizes the famous Matsumoto's slope-of-a-mountain problem (in Part II).

Part I. Different aspects of complex Finsler geometry

Professor Shiing-Shen Chern wrote [70]:

> *"Complex Finsler geometry is extremely beautiful. Again the bundle of line elements PTM plays the important role. The scalar product on the pulled-back TM gives rise to a Hermitian structure on the complexification of the latter. Here the geometrical properties mix well with the complex structure; connection forms are of type $(1,0)$ and curvature forms are of type $(1,1)$. A real valued holomorphic curvature, as a function on PTM, can be introduced".* [1]

Bearing in mind the perspective of S. S. Chern, we tried to introduce general themes from real Finsler geometry into complex Finsler geometry. Nevertheless, there are noteworthy differences and specific tools when compared to the real reasoning, all these being emphasized in each chapter of Part I. Certainly, as expected, our study is far from being complete. Here we point out just some of the main problems that we have approached in complex Finsler geometry, mostly based on our papers [24, 25, 26, 27, 36, 23, 40, 14]. More precisely, in this part, we present the complex Landsberg and generalized Berwald spaces as well as a few particular cases of complex Landsberg spaces for which the unicorn problem has no solution (Chapter 2). The problem of the projectively related complex Finsler metrics is addressed

[1] Shiing-Shen Chern, *Finsler Geometry Is Just Riemannian Geometry without the Quadratic Restriction*, Notices of the AMS, September 1996.

in Chapter 3, mainly focused on the complex versions of Rapcsák's theorem and a complex Finsler solution of Hilbert's fourth problem. Making full use of the projective changes of the complex Finsler metrics, in Chapter 4 we present a few projective curvature invariants of Douglas and Weyl type which then allow us to point out some complex Finsler spaces of constant holomorphic curvature along with the complex Douglas spaces. At the end of this part, in Chapter 5 we consider Zermelo's navigation problem on a Hermitian manifold, trying to indicate how some properties of a Hermitian metric are affected by the Zermelo deformation under action of some special winds.

There is not enough space here to prove with other results that our contribution in complex Finsler field is relevant (see for example: [31, 32, 34] for complex Cartan spaces, [38, 37, 28] for η-Einstein spaces and holomorphic sectional and bisectional curvatures, [115, 29, 30] for a complex Finsler approach of gravity, [14, 15] for complex Finsler solutions of Zermelo navigation problem on Hermitian manifolds). However, from our point of view the presentation of this part seems natural.

Part II. Extensions of Matsumoto's slope-of-a-mountain problem

Professor Paul Finsler answered in 1969 [106], [107, §16]:

> *"In the astronomy we measure the distance in a time, in particular, in the light-year. When we take a second as the unit, the unit surface is a sphere with the radius of 300,000 km. To each point of our space is associated such a sphere; this defines the distance (measured in a time) and the geometry of our space is the simplest one, namely, the euclidean geometry. Next, when a ray of light is considered as the shortest line in the gravitational field, the geometry of our space is Riemannian geometry. Furthermore, in an anisotropic medium the speed of the light depends on its direction, and the unit surface is not any longer a sphere. Now, on a slope of the earth surface we sometimes measure the distance in a time, namely, the time required such as seen on a guidepost. Then the unit curve, taken a minute as the unit, will be a general closed curve without centre, because we can walk only a shorter distance in an uphill road than in a downhill road. This defines a general geometry, although it is not exact. The shortest line along which we can reach the goal, for instance, the top of a mountain as soon as possible will be a complicated curve."* [2]

The idea for the results presented in this part comes from two iconic problems: Zermelo's navigation problem (ZNP) and Matsumoto's slope-of-a-mountain problem (MAT) which are continuously topical issues and examples in the research area of Riemann-Finsler geometry because of their intuitively clear formulation, modelling and significant applications in physics. The former, which was stated originally much earlier and solved by Ernst Zermelo, was intended initially to determine the shortest time paths of an object that moves at a constant self-speed on the Euclidean plane in the presence of an external force like wind or water current [157]. In recent years, the problem was generalized substantially and considered on Riemannian manifolds of arbitrary dimension in purely geometric formulation in the presence of weak vector fields [127, 45, 71, 124]. Furthermore, the study was extended for stronger winds including some investigations on global solutions [154, 61]. The latter was investigated

[2] The answer of P. Finsler to the question of M. Matsumoto regarding "models of the Finsler spaces".

for the first time by Makoto Matsumoto (inspired by the answer of P. Finsler [106], [107, §16]) and also refers to the fastest trajectories (time geodesics) of a person who walks or runs on a mountain slope under the influence of gravity, taking into consideration that walking uphill is more tiring than walking downhill (see also [62, 63, 158]). The corresponding research on the issues in weak vector fields has led to the solutions provided by special Finsler metrics, namely, of Randers and Matsumoto type, respectively. It is worth pointing out that both issues are presented in literature as two different problems (see for example [44, Sec. 1.1.1]), which thus far have been described and studied separately, although their solutions belong to the class of (α, β)-metrics. In particular, the geometric construction of an indicatrix based on a rigid translation of a background Riemannian metric in Zermelo's navigation problem differs considerably from a direction-dependent deformation included in the Matsumoto problem. This fact is related directly to the essential difference in the equations of motion that underlie ZNP and MAT.

The general aim of Part II, focused on our results obtained in [10, 20, 11, 12, 13], is to present a set of navigation problems on a slippery mountain slope that is a Riemannian manifold under the action of some active winds, expressed in terms of a gravitational wind and two traction coefficients (separately or even together). A crucial role in our study is played by the gravitational wind, which lets us collect and describe all time-optimal problems mentioned in this part, including the classical ones (ZNP and MAT), in a convenient, unified and effective manner. The key aspect we want to emphasize is the type and range of compensation of the gravity effects in the described models of the mountain slopes, which then characterize the general equations of motion and, consequently, the related Finsler metric in each case. Our study generalizes Matsumoto's initial exposition [106], whilst, at the same time, creating a direct link between MAT and ZNP. In fact, these two classical problems in Finsler geometry become the particular and boundary cases of our study. The investigation presented also provides new applications of the respective general (α, β)-metric that are described in this work. More precisely, we formulate and solve the slippery slope problem (Chapter 7) and the slippery-cross-slope problem that includes the navigation problem under the cross gravitational wind (Chapter 8). Chapter 9 unifies and extends all navigation problems developed in the previous chapters, by the most general model of a slippery mountain slope. The key detail is that here both the transverse and longitudinal gravity-additives with respect to direction of motion are admitted to vary simultaneously in full ranges.

Part I. Different aspects of complex Finsler geometry

Chapter 1

Rudiments of complex Finsler geometry

This chapter briefly recalls some basic notions about complex Finsler geometry with a few of its tools (e.g., Chern-Finsler, Berwald and Rund complex linear connections) that are needed for presenting the next chapters. For more details we specify [1, 116, 5, 6]. In particular, we focus on some important properties of the aforementioned complex linear connections that were proved in our paper [26].

1.1 Complex Finsler spaces

Let M be an n-dimensional complex manifold, $z = (z^k)_{k=\overline{1,n}}$ be the complex coordinates in a local chart. Note that by $k = \overline{1,n}$ we mean $k = 1, ..., n$.

The complexified $T_{\mathbb{C}}M$ of the real tangent bundle $T_{\mathbb{R}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ ($\pi : T'M \to M$) is itself a complex manifold and the local coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=\overline{1,n}}$. These are changed into $(z'^k, \eta'^k)_{k=\overline{1,n}}$ by the rules

$$z'^k = z'^k\,(z) \quad \text{and } \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j. \tag{1.1}$$

A *complex Finsler space* is a pair (M, F), where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:
i) $L = F^2$ is smooth on $\widetilde{T'M} = T'M \backslash \{0\}$;
ii) $F(z, \eta) \geq 0$ for all $(z, \eta) \in T'M$; the equality holds if and only if $\eta = 0$;
iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for all $(z, \eta) \in T'M$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$;
iv) the Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$ is positive definite, where $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor. Equivalently, this means that the indicatrix of F is *strongly pseudoconvex.*

A function f on $T'M$ is called (p, q)-*homogeneous* with respect to η and $\bar{\eta}$, respectively if $f(z, \lambda\eta) = \lambda^p \bar{\lambda}^q f(z, \eta)$, for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$. By Euler's theorem, this homogeneity condition is equivalent to $\frac{\partial f}{\partial \eta^k}\eta^k = pf$ and $\frac{\partial f}{\partial \bar{\eta}^k}\bar{\eta}^k = qf$. Consequently, from iii) we have

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L, \quad \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i\bar{\eta}^j.$$

Thus, $L = F^2$ is $(1,1)$-homogeneous and $g_{i\bar{j}}(z,\eta)$ are $(0,0)$-homogeneous with respect to η and $\bar{\eta}$, respectively.

Roughly speaking, the geometry of a complex Finsler space consists of the study of the geometric objects of the complex manifold $T'M$ endowed with the Hermitian metric structure defined by $g_{i\bar{j}}$. Therefore, the first step is to study the sections of the complexified $T_{\mathbb{C}}(T'M)$ of the real tangent bundle $T_{\mathbb{R}}(T'M)$, which is decomposed in the sum

$$T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M).$$

Let $VT'M = \ker \pi_* \subset T'(T'M)$ be the vertical sub-bundle, locally spanned by $\{\dot{\partial}_k = \frac{\partial}{\partial \eta^k}\}$, and $VT''M$ be its conjugate. A natural local frame for $T'_u(T'M)$ is $\{\frac{\partial}{\partial z^k}, \dot{\partial}_k\}$ and the Jacobi matrix of the above transformations (1.1) gives the changinig rules for $\frac{\partial}{\partial z^k}$ and $\dot{\partial}_k$. A complicate form of the change rule for $\frac{\partial}{\partial z^k}$ leads to idea of *complex nonlinear connection*, briefly (*c.n.c.*), which is a tool "to linearise" this geometry. More precisely, a (*c.n.c.*) refers to the horizontal sub-bundle $HT'M$ in $T'(T'M)$, such that $T'(T'M) = HT'M \oplus VT'M$ and $H_uT'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*), that hold the certain rule

$$N'^i_j \frac{\partial z'^j}{\partial z^k} = \frac{\partial z'^i}{\partial z^j} N_k^j - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^j. \tag{1.2}$$

The pair $\{\delta_k = \frac{\delta}{\delta z^k}, \dot{\partial}_k = \frac{\partial}{\partial \eta^k}\}$ is called the adapted frame of the (*c.n.c.*) which obey the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere, it results an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted frames are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

A section on $T'(T'M)$, locally expressed as follows

$$S = \eta^k \frac{\partial}{\partial z^k} - 2G^k(z,\eta)\frac{\partial}{\partial \eta^k}, \tag{1.3}$$

is a *complex spray*, where G^k denote the spray coefficients (see [116]). Under the changes of complex coordinates on $T'M$, the coefficients G^k of the spray S hold the rule

$$2G'^i = 2G^k \frac{\partial z'^i}{\partial z^k} - \frac{\partial^2 z'^i}{\partial z^j \partial z^k}\eta^j \eta^k. \tag{1.4}$$

The notions of complex spray and (*c.n.c.*) are interdependent, one determining the other. Differentiating (1.4) with respect to η^j, it follows that the functions $N_j^i = \frac{\partial G^i}{\partial \eta^j}$ satisfy the rule (1.2), and hence N_j^i define a nonlinear connection. Conversely, any (*c.n.c.*) determines a complex spray. Indeed, a simple computation shows that if N_j^i are the coefficients of a (*c.n.c.*), then $\frac{1}{2}N_j^i\,\eta^j$ satisfy (1.4) and thus, they define a complex spray.

Certainly, a main problem in this geometry is to determine a (*c.n.c.*) related only to the fundamental function of the complex Finsler space (M,F) and corresponding to it the action of a derivative law D on the sections of $T_{\mathbb{C}}(T'M)$. A well-known solution is provided by Chern-Finsler (*c.n.c.*), with the local coefficients

$$N_j^i = g^{\overline{m}i}\frac{\partial g_{l\overline{m}}}{\partial z^j}\eta^l,$$

which are $(1,0)$-homogeneous with respect to η and $\bar{\eta}$, respectively $((\dot{\partial}_k N_j^i)\eta^k = N_j^i$ and $(\dot{\partial}_{\bar{k}} N_j^i)\bar{\eta}^k = 0)$, being a main tool in complex Finsler geometry (see [116]). From now on, by δ_k we mean the adapted frame with respect to the Chern-Finsler (*c.n.c.*).

Corresponding to Chern-Finsler (*c.n.c.*), there exists a good complex vertical connection D called Chern-Finsler connection in [1] or Hermitian-Finsler connection in [5, 6]. This means that it is of $(1,0)$-type (i.e. $D_{JX}Y = JD_XY$, for any section X on $T'(T'M)$ and for any vertical vector field Y, where J is the natural complex structure on $T_{\mathbb{C}}M$) and metrical with respect to the Hermitian structure. Following the notations from [116], the Chern-Finsler connection is locally given by $CF\Gamma = (N^i_j, L^i_{jk}, L^{\bar{\imath}}_{\bar{\jmath}k}, C^i_{jk}, C^{\bar{\imath}}_{\bar{\jmath}k})$, where

$$N^i_j = L^i_{lj}\eta^l, \quad L^i_{jk} = g^{\bar{l}i}\delta_k g_{j\bar{l}} = \dot{\partial}_j N^i_k, \quad C^i_{jk} = g^{\bar{l}i}\dot{\partial}_k g_{j\bar{l}}, \quad L^{\bar{\imath}}_{\bar{\jmath}k} = C^{\bar{\imath}}_{\bar{\jmath}k} = 0 \tag{1.5}$$

and $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$, $D_{\delta_k}\delta_{\bar{\jmath}} = L^{\bar{\imath}}_{\bar{\jmath}k}\delta_{\bar{\imath}}$, $D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$, $D_{\dot{\partial}_k}\dot{\partial}_{\bar{\jmath}} = C^{\bar{\imath}}_{\bar{\jmath}k}\dot{\partial}_{\bar{\imath}}$.

Denoting by " $_{|}$ " , " $|$ " , " $\bar{|}$ " and "$\bar{|}$", the h-, v-, $\bar{h}$-, $\bar{v}$- covariant derivatives with respect to Chern-Finsler connection, respectively, it turns out the following relations

$$\begin{aligned} \eta^i_{|k} = \eta^i_{|\bar{k}} = \eta^i|_{\bar{k}} = 0, \quad \eta^i|_k = \delta^i_k, \\ g_{i\bar{\jmath}|k} = g_{i\bar{\jmath}|\bar{k}} = g_{i\bar{\jmath}}|_k = g_{i\bar{\jmath}}|_{\bar{k}} = 0. \end{aligned} \tag{1.6}$$

Now, we consider the complex Cartan tensors: $C_{i\bar{\jmath}k} = \dot{\partial}_k g_{i\bar{\jmath}}$ and $C_{i\bar{\jmath}\bar{k}} = \dot{\partial}_{\bar{k}} g_{i\bar{\jmath}}$.

Lemma 1.1.1. *For any complex Finsler space (M, F), the following statements hold:*
i) $C_{l\bar{r}h|k} = (\dot{\partial}_h L^i_{lk})g_{i\bar{r}}$;
ii) $C_{l\bar{r}\bar{h}|k} = (\dot{\partial}_{\bar{h}} L^i_{lk})g_{i\bar{r}} + (\dot{\partial}_{\bar{h}} N^i_k)C_{i\bar{r}l}$.

Proof. Differentiating $N^i_k g_{i\bar{r}} = \frac{\partial g_{j\bar{r}}}{\partial z^k}\eta^j$ with respect to η^l, this gives

$$L^i_{lk}g_{i\bar{r}} = \frac{\partial g_{l\bar{r}}}{\partial z^k} - N^i_k C_{i\bar{r}l}. \tag{1.7}$$

Now, differentiating in (1.7) with respect to η^h it results i), and then with respect to $\bar{\eta}^h$, it leads to ii). □

Recall that $R^i_{j\bar{h}k} = -\delta_{\bar{h}}L^i_{jk} - (\delta_{\bar{h}}N^l_k)C^i_{jl}$ denote the $h\bar{h}$ - curvatures coefficients of Chern-Finsler connection. According to [1, p. 108] and [116, p. 81], the *holomorphic curvature* of the complex Finsler space (M, F) in direction η is defined by

$$\mathcal{K}_F(z,\eta) = \frac{2}{L^2}R_{\bar{r}j\bar{k}h}\bar{\eta}^r\eta^j\bar{\eta}^k\eta^h, \tag{1.8}$$

where $R_{\bar{r}j\bar{k}h} = R^i_{j\bar{k}h}g_{i\bar{r}}$.

In [1]'s terminology, the complex Finsler space (M, F) is *strongly Kähler* iff $T^i_{jk} = 0$, *Kähler* iff $T^i_{jk}\eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}}T^i_{jk}\eta^j\bar{\eta}^l = 0$, where $T^i_{jk} = L^i_{jk} - L^i_{kj}$. In [65] it is proved that strongly Kähler and Kähler notions actually coincide. We note that in the particular case of the complex Finsler metrics which come from Hermitian metrics on M, so-called *pure Hermitian metrics* in [116] (i.e. $g_{i\bar{\jmath}} = g_{i\bar{\jmath}}(z)$), all these types of Kähler coincide.

1.2 Connections on a complex Finsler space

As already emphasized in the previous section, between a complex spray and a (*c.n.c.*) there exists an interdependence, one determining the other. In [116] it is proved that the Chern-Finsler (*c.n.c.*) does not generally come from a complex spray, excepting the case when the complex metric is Kähler. On the other hand, its local coefficients $N_j^k = g^{\bar{m}k}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l$ always determine a complex spray with the coefficients $G^i = \frac{1}{2}N_j^i\eta^j$, which are $(2,0)$-homogeneous with respect to η and $\bar{\eta}$, respectively $((\dot{\partial}_k G^i)\eta^k = 2G^i$ and $(\dot{\partial}_{\bar{k}}G^i)\bar{\eta}^k = 0)$. Furthermore, G^i induce a (*c.n.c.*) with the local coefficients denoted by $\overset{c}{N}{}_j^i = \dot{\partial}_j G^i$ and called *canonical* in [116], where it is proved that the canonical (*c.n.c.*) coincides with Chern-Finsler (*c.n.c.*) if and only if the complex Finsler metric is Kähler.

Further on, we consider the frame $\{\overset{c}{\delta}_k, \dot{\partial}_k\}$ with respect to the canonical (*c.n.c.*), where $\overset{c}{\delta}_k = \frac{\partial}{\partial z^k} - \overset{c}{N}{}_k^j\dot{\partial}_j$, as well as the dual frame $\{dz^k, \overset{c}{\delta}\eta^k\}$, where $\overset{c}{\delta}\eta^k = d\eta^k + \overset{c}{N}{}_j^k dz^j$. Moreover, we associate to the canonical (*c.n.c.*) two complex linear connections. One is of Berwald type $B\Gamma = (\overset{c}{N}{}_j^i, G^i_{jk}, G^i_{j\bar{k}}, 0, 0)$ having the connection form

$$\omega_j^i(z,\eta) = G^i_{jk}dz^k + G^i_{j\bar{k}}d\bar{z}^k, \tag{1.9}$$

where $G^i_{jk} = \dot{\partial}_k\overset{c}{N}{}_j^i = G^i_{kj}$ and $G^i_{j\bar{k}} = \dot{\partial}_{\bar{k}}\overset{c}{N}{}_j^i$. Another is a complex linear connection of Rund type $R\Gamma = (\overset{c}{N}{}_j^i, \overset{c}{L}{}^i_{jk}, \overset{c}{L}{}^i_{j\bar{k}}, 0, 0)$, where $\overset{c}{L}{}^i_{jk} = \frac{1}{2}g^{\bar{l}i}(\overset{c}{\delta}_k g_{j\bar{l}} + \overset{c}{\delta}_j g_{k\bar{l}})$ and $\overset{c}{L}{}^i_{j\bar{k}} = \frac{1}{2}g^{\bar{l}i}(\overset{c}{\delta}_{\bar{k}} g_{j\bar{l}} - \overset{c}{\delta}_{\bar{l}} g_{j\bar{k}})$. We note that $R\Gamma$ is only h-metrical and $B\Gamma$ is neither h- nor v-metrical (for more details see [116]) and the spray coefficients hold the relations $2G^i = N^i_j\eta^j = \overset{c}{N}{}^i_j\eta^j = G^i_{jk}\eta^j\eta^k$ and $\overset{c}{\delta}_j = \delta_j - (\overset{c}{N}{}^k_j - N^k_j)\dot{\partial}_k$.

A few additional properties are specific to the Kähler case. Namely, under Kähler assumption one has that $\overset{c}{\delta}_j = \delta_j$ ([116], p. 68) and thus, $\overset{c}{\delta}_j g_{k\bar{h}} = \overset{c}{\delta}_k g_{j\bar{h}}$. If we contract the last equality with $g^{\bar{l}j}$ it results $g^{\bar{l}j}(\overset{c}{\delta}_j g_{k\bar{h}} - \overset{c}{\delta}_k g_{j\bar{h}}) = 0$, that is $\overset{c}{L}{}^{\bar{l}}_{\bar{h}j} = 0$. By conjugation, it follows that $\overset{c}{L}{}^i_{h\bar{j}} = 0$. Also, in the Kähler case one has $L^i_{jk} = \overset{c}{L}{}^i_{jk} = G^i_{jk}$.

Further on, the tools related to the Berwald and Rund connections will be specified everywhere by a centred superscript, like above (e.g. $\overset{c}{\delta}_k$, $\overset{c}{L}{}^i_{jk}$, $X_{\overset{B}{|k}}$, etc.), while for the Chern-Finsler connection we keep the initial generic notation, without centred superscript (e.g. δ_k, L^i_{jk}, $X_{|k}$, etc.).

Lemma 1.2.1. *For any complex Finsler space (M,F), the following statements hold:*
i) $G^i_{j\bar{k}}\bar{\eta}^k = 0$;
ii) $g_{l\bar{r}\overset{B}{|}h} + g_{h\bar{r}\overset{B}{|}l} + G^{\bar{m}}_{\bar{r}h}g_{l\bar{m}} + G^{\bar{m}}_{\bar{r}l}g_{h\bar{m}} = -C_{l\bar{r}h\overset{B}{|}0}$;
iii) $2(\dot{\partial}_{\bar{h}}G^i)g_{i\bar{r}} = C_{0\bar{r}\bar{h}\overset{B}{|}0} = C_{0\bar{r}\bar{h}|0}$;
iv) $C_{i\bar{j}h\overset{B}{|}k} = \dot{\partial}_h(g_{i\bar{j}\overset{B}{|}k}) + (\dot{\partial}_h G^l_{ik})g_{l\bar{j}} + (\dot{\partial}_h G^{\bar{m}}_{\bar{j}k})g_{i\bar{m}}$;
v) $C_{i\bar{r}\bar{h}\overset{B}{|}k} = \dot{\partial}_{\bar{h}}(g_{i\bar{j}\overset{B}{|}k}) + (\dot{\partial}_{\bar{h}}G^l_{ik})g_{l\bar{j}} + (\dot{\partial}_{\bar{h}}G^{\bar{m}}_{\bar{j}k})g_{i\bar{m}} + G^l_{k\bar{h}}C_{i\bar{j}l} - G^{\bar{m}}_{\bar{h}k}C_{i\bar{j}\bar{m}}$,

where the index 0 *means the contraction by the fiber coordinate* η *and* $\overset{B}{|}$ *is h-covariant derivative with respect to* $B\Gamma$.

Proof. i) $G^i_{j\bar{k}}\bar{\eta}^k = \dot{\partial}_j[(\dot{\partial}_{\bar{k}}G^i)\bar{\eta}^k] = \frac{1}{2}\dot{\partial}_j[(\dot{\partial}_{\bar{k}}N^i_l)\eta^l\bar{\eta}^k] = \frac{1}{2}\dot{\partial}_j[\dot{\partial}_{\bar{k}}(g^{\bar{m}i}\frac{\partial g_{s\bar{m}}}{\partial z^l}\eta^s)\bar{\eta}^k\eta^l] = 0.$
ii) $G^i = \frac{1}{2}N^i_j\eta^j = \frac{1}{2}g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial z^h}\eta^j\eta^k$ can be rewritten as follows

$$G^i g_{i\bar{r}} = \frac{1}{2}\frac{\partial g_{j\bar{r}}}{\partial z^k}\eta^j\eta^k. \tag{1.10}$$

Differentiating (1.10) with respect to η^l, it yields

$$\overset{c}{N}{}^i_l g_{i\bar{r}} = \frac{1}{2}\left(\frac{\partial g_{l\bar{r}}}{\partial z^k} + \frac{\partial g_{k\bar{r}}}{\partial z^l}\right)\eta^k - G^i C_{i\bar{r}l}. \tag{1.11}$$

Differentiating (1.11) with respect to η^h it results

$$G^i_{hl}g_{i\bar{r}} = \frac{1}{2}\left(\frac{\partial g_{l\bar{r}}}{\partial z^h} + \frac{\partial g_{h\bar{r}}}{\partial z^l}\right) + \frac{1}{2}\frac{\partial C_{l\bar{r}h}}{\partial z^k}\eta^k - G^i(\dot{\partial}_h C_{i\bar{r}l}) - \overset{c}{N}{}^i_h C_{i\bar{r}l} - \overset{c}{N}{}^i_l C_{i\bar{r}h}, \tag{1.12}$$

which leads to

$$-(\overset{c}{\delta}_k C_{l\bar{r}h})\eta^k + \overset{c}{N}{}^i_h C_{i\bar{r}l} + \overset{c}{N}{}^i_l C_{i\bar{r}h} = \overset{c}{\delta}_h g_{l\bar{r}} + \overset{c}{\delta}_l g_{h\bar{r}} - 2G^i_{hl}g_{i\bar{r}}. \tag{1.13}$$

Now, taking into account that $g_{l\bar{r}\overset{B}{|}h} = \overset{c}{\delta}_h g_{l\bar{r}} - G^i_{lh}g_{i\bar{r}} - G^{\bar{m}}_{\bar{r}h}g_{l\bar{m}}$ and i) it turns out ii).
iii) Differentiating (1.10) with respect to $\bar{\eta}^h$, it turns out that

$$2(\dot{\partial}_{\bar{h}}G^i)g_{i\bar{r}} = \frac{\partial C_{l\bar{r}\bar{h}}}{\partial z^k}\eta^l\eta^k - \overset{c}{N}{}^i_k\eta^k C_{i\bar{r}\bar{h}}. \tag{1.14}$$

But, one has also $\dot{\partial}_l(C_{j\bar{s}\bar{h}}\eta^j) = (\dot{\partial}_l C_{j\bar{s}\bar{h}})\eta^j + C_{l\bar{s}\bar{h}} = \dot{\partial}_{\bar{h}}(C_{j\bar{s}l}\eta^j) + C_{l\bar{s}\bar{h}} = C_{l\bar{s}\bar{h}}$. Thus, using (1.14) and $N^i_j\eta^j = \overset{c}{N}{}^i_j\eta^j$ we obtain $2(\dot{\partial}_{\bar{h}}G^i)g_{i\bar{r}} = \overset{c}{\delta}_k(C_{l\bar{r}\bar{h}}\eta^l)\eta^k = \delta_k(C_{l\bar{r}\bar{h}}\eta^l)\eta^k$, which together with i) and the h-covariant derivative rule with respect to Chern-Finsler connection gives iii).
iv) Using again $g_{i\bar{j}\overset{B}{|}k} = \overset{c}{\delta}_k g_{i\bar{j}} - G^l_{ik}g_{l\bar{j}} - G^{\bar{m}}_{\bar{j}k}g_{i\bar{m}}$, which differentiated now with respect to η^h, gives

$$\dot{\partial}_h(g_{i\bar{j}\overset{B}{|}k}) = \frac{\partial C_{i\bar{j}h}}{\partial z^k} - G^l_{hk}C_{i\bar{j}l} - \overset{c}{N}{}^l_k(\dot{\partial}_h C_{i\bar{j}l}) - (\dot{\partial}_h G^l_{ik})g_{l\bar{j}} - G^l_{ik}C_{l\bar{j}h} - (\dot{\partial}_h G^{\bar{m}}_{\bar{j}k})g_{i\bar{m}} - G^{\bar{p}}_{\bar{j}k}C_{i\bar{m}h}$$

$$= \overset{c}{\delta}_k C_{i\bar{j}h} - G^l_{hk}C_{i\bar{j}l} - G^l_{ik}C_{l\bar{j}h} - G^{\bar{m}}_{\bar{j}k}C_{i\bar{m}h} - (\dot{\partial}_h G^l_{ik})g_{l\bar{j}} - (\dot{\partial}_h G^{\bar{m}}_{\bar{j}k})g_{i\bar{m}}$$

$$= C_{i\bar{j}h\overset{B}{|}k} - (\dot{\partial}_h G^l_{ik})g_{l\bar{j}} - (\dot{\partial}_h G^{\bar{m}}_{\bar{j}k})g_{i\bar{m}},$$

that is iv).

For v) we compute

$$\dot{\partial}_{\bar{h}}(g_{i\bar{j}\,\overset{B}{|}\,k}) = \frac{\partial C_{i\bar{j}\bar{h}}}{\partial z^k} - G^l_{k\bar{h}} C_{i\bar{j}l} - N^l_k(\dot{\partial}_l C_{i\bar{j}\bar{h}}) - (\dot{\partial}_{\bar{h}} G^l_{ik}) g_{l\bar{j}} - G^l_{ik} C_{l\bar{j}\bar{h}} - (\dot{\partial}_{\bar{h}} G^{\bar{m}}_{\bar{j}k}) g_{i\bar{m}} - G^{\bar{m}}_{\bar{j}k} C_{i\bar{p}\bar{m}}$$

$$= \delta_k C_{i\bar{j}\bar{h}} - G^l_{ik} C_{l\bar{j}\bar{h}} - G^{\bar{m}}_{\bar{h}k} C_{i\bar{j}\bar{m}} - G^{\bar{m}}_{\bar{j}k} C_{i\bar{m}\bar{h}} - G^l_{k\bar{h}} C_{i\bar{j}l}$$

$$+ G^{\bar{m}}_{\bar{h}k} C_{i\bar{j}\bar{m}} - (\dot{\partial}_{\bar{h}} G^l_{ik}) g_{l\bar{j}} - (\dot{\partial}_{\bar{h}} G^{\bar{m}}_{\bar{j}k}) g_{i\bar{m}}$$

$$= C_{i\bar{j}\bar{h}\,\overset{B}{|}\,k} - G^l_{k\bar{h}} C_{i\bar{j}l} + G^{\bar{m}}_{\bar{h}k} C_{i\bar{j}\bar{m}} - (\dot{\partial}_{\bar{h}} G^l_{ik}) g_{l\bar{j}} - (\dot{\partial}_{\bar{h}} G^{\bar{m}}_{\bar{j}k}) g_{i\bar{m}}.$$

□

Chapter 2

On complex Landsberg spaces

This chapter presents the concepts of complex Landsberg and generalized Berwald spaces that were first introduced by us in [26], as well as a few of their subclasses. More precisely, the intersection of these two sets of complex Finsler spaces provides the set of G-Landsberg spaces that includes two other, strong Landsberg and G-Kähler spaces. We prove that a G-Kähler space coincides with a Kähler-Berwald space and it is included in the set of the strong Landsberg spaces. Some special complex Finsler spaces with (α, β)-metrics (introduced by us in [27, 36]) offer examples of generalized Berwald spaces.

2.1 Introduction and the main results

The real Landsberg spaces, in particular the real Berwald spaces, have been a major subject of study for many geometers over the years. In 1926 L. Berwald introduced a special class of Finsler spaces which took his name in 1964. It is known that a real Finsler space is called a *Berwald space* if the local coefficients of the Berwald connection depend only on position coordinates. An equivalent condition to this is that the Cartan tensor field is h-parallel to the Berwald connection, (i.e. $C_{ijk;r} = 0$, where ";" means the horizontal covariant derivative with respect to the Berwald connection). In 1934 É. Cartan emphasized that the Berwald connection is not metrical and $g_{ij;k} = -2C_{ijk;0}$. Therefore if $C_{ijk;0} = 0$, then it becomes metrical. However, such a space was called a *Landsberg space* by L. Berwald in 1928.

Many great contributions to the geometry of the real Landsberg and Berwald spaces have been made by Z. Szabo [136], M. Matsumoto [108], P. Antonelli [42], A. Bejancu [50], Z. Shen [128], etc. Every Berwald space is a Landsberg space. The converse, has been a long-standing problem [100, 137, 73].

Some general themes from real Finsler geometry about Landsberg and Berwald spaces were broached in complex Finsler geometry by us (see [26]). There are noteworthy differences compared to real reasoning, mainly on account of the fact that in complex Finsler geometry there are two different horizontal covariant derivatives, in particular for the Cartan tensors, one has $C_{i\bar{j}k|h}$ and the other $C_{i\bar{j}k|\bar{h}}$ with respect to Chern-Finsler connection. As we have already proved by Lemma 2.2.15, for any complex Finsler space, the condition $C_{l\bar{r}h|k} = 0$ is equivalent to $C_{l\bar{r}h|\bar{k}} = 0$ and moreover, the horizontal coefficients of the Chern-Finsler connection depend only on the position coordinates, namely $L^i_{jk}(z)$, in this case. Perhaps, this reason led T. Aikou to call the complex Finsler spaces with $L^i_{jk}(z)$, complex Berwald spaces [7]. However, the condition for complex Berwald space can be characterized by the fact that the horizontal coefficients G^i_{jk} of the complex linear connection of Berwald type $B\Gamma$ are independent of the fibre coordinates, only in the particular context of Kähler assumption when, $L^i_{jk} = G^i_{jk}$. Therefore an unquestionable extension of the complex Berwald spaces, directly related to the linear connection $B\Gamma$, is provided by a generalized Berwald space characterized by the fact that G^i_{jk} depend only on the position z. Some characteristics of the generalized Berwald space are collected in Theorem 2.2.18. We note that an interesting study of the generalized Berwald spaces was also done by C. Zhong in [160], where the terminology *weakly complex Berwald* spaces is used for these.

The same arguments as in the real case were taken into account to define a complex Landsberg space in [26]. Since in the real case, a Finsler space is Landsberg if the Berwald and Rund connections coincide, we also used as a toolkit, besides $B\Gamma$, another complex linear connection of Rund type $R\Gamma$, both associated to the canonical complex nonlinear connection, with the local coefficients $\overset{c}{N}{}^i_j = \dot{\partial}_j G^i$. However, in complex Finsler geometry the things are considerably more difficult. On one hand, the connections $B\Gamma$ and $R\Gamma$ are in general not of $(1,0)$-type as Chern-Finsler connection. On the other hand, in the complex case alongside the horizontal covariant derivative with respect to $B\Gamma$, we also have its conjugate and thus it is hardly to control the relationships between these. Here, we speak about complex *Landsberg* space iff $G^i_{jk} = \overset{c}{L}{}^i_{jk}$ and various characterizations of the complex Landsberg spaces are provided by Theorem 2.2.2. Further on, we have defined the class of *G-Landsberg* spaces. This is included in the class of complex Landsberg spaces with $\dot{\partial}_{\bar{k}} G^i = 0$. Theorem 2.2.8 reports on the necessary and sufficient conditions for a complex Finsler space to be a G-Landsberg space. A reinforcement of the tensorial characterization for a G-Landsberg space gives rise to a subclass of G-Landsberg, namely a complex Finsler space is *strong Landsberg* iff $C_{l\bar{r}h\overset{B}{|}0} = 0$ and $C_{j\bar{r}h\overset{B}{|}\bar{0}} = 0$. Other characteristics of the strong Landsberg spaces are combined in Theorem 2.2.10. Because any Kähler space is a complex Landsberg space, the substitution of the Landsberg condition with the Kähler condition in the definition of a G-Landsberg space had to lead in [26] to another subclass of this, called *G-Kähler.* Among other things, Theorem 2.2.13 provides that a G-Kähler space coincides with a complex Berwald space that satisfies in addition the Kähler condition (a *Kähler-Berwald space*). The strong Landsberg spaces are situated somewhere between complex Berwald spaces and G-Landsberg spaces. The proof of the aforementioned theorems is presented in Section 2.2 as well as the interrelations among all these classes of complex Finsler spaces. An intuitive scheme with all these spaces is summarized in Figure 2.1.

The general theory on generalized Berwald spaces is fulfilled by some special outcomes in [26] for the class of complex Finsler spaces with (α, β)-metrics (see Section 2.3). More precisely, we prove that a complex Randers space assumed to be a generalized Berwald and the weakly Kähler is Kähler-Berwald (Theorem 2.3.6).

2.2 From complex Landsberg to generalized Berwald spaces

We begin by pointing out a few complex Finsler spaces of Landsberg type.

Definition 2.2.1. *Let (M, F) be an n-dimensional complex Finsler space. (M, F) is called complex Landsberg space if $G^i_{jk} = \overset{c}{L}{}^i_{jk}$.*

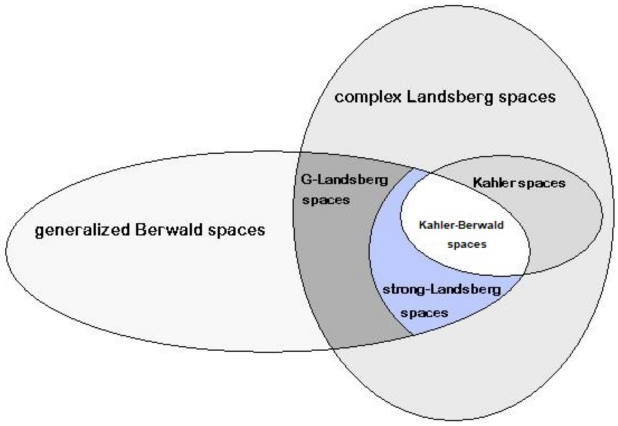


Figure 2.1: Inclusions

It is worthwhile to mention that any complex Finsler space that is Kähler is a Landsberg space, because under Kähler assumption, $L^i_{jk} = \overset{c}{L}{}^i_{jk} = G^i_{jk}$. Therefore, the Kähler spaces offer an asset family of complex Landsberg spaces.

Theorem 2.2.2. *Let (M, F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:*
i) (M, F) is a complex Landsberg space;
ii) $C_{l\bar{r}h\overset{B}{|}0} = 0$;
iii) $2(\dot{\partial}_h G^i_{jk})g_{i\bar{r}} - G^{\bar{m}}_{\bar{r}k} C_{j\bar{m}h} - G^{\bar{m}}_{\bar{r}j} C_{k\bar{m}h} = C_{j\bar{r}h\overset{B}{|}k} + C_{k\bar{r}h\overset{B}{|}j}$;
iv) $g_{i\bar{j}\overset{B}{|}k} = (\overset{c}{L}{}^{\bar{m}}_{\bar{j}k} - G^{\bar{m}}_{\bar{j}k})g_{i\bar{m}}$.

Proof. i) $\Leftrightarrow$ ii). A direct computation gives the relation

$$g_{l\bar{r}\overset{B}{|}h} + g_{h\bar{r}\overset{B}{|}l} + G^{\bar{m}}_{\bar{r}h} g_{l\bar{m}} + G^{\bar{m}}_{\bar{r}l} g_{h\bar{m}} = 2(\overset{c}{L}{}^i_{lh} g_{i\bar{r}} - G^i_{lh} g_{i\bar{r}})$$

which, together with Lemma 1.2.1 ii), provides this equivalence.
i) $\Rightarrow$ iii). Since (M, F) is Landsberg, one has that $G^i_{jk} g_{i\bar{r}} = \frac{1}{2}(\overset{c}{\delta}_k g_{j\bar{r}} + \overset{c}{\delta}_j g_{k\bar{r}})$. Differentiating it with respect to η^h yields iii).
iii) $\Rightarrow$ ii). By contracting in iii) with η^k, it turns out that $2(\dot{\partial}_h G^i_{jk}) g_{i\bar{r}} \eta^k = C_{j\bar{r}h\overset{B}{|}0}$. On the other hand, $(\dot{\partial}_h G^i_{jk}) g_{i\bar{r}} \eta^k = 0$. From this we obtain ii).
i) $\Leftrightarrow$ iv). $g_{i\bar{j}\overset{B}{|}k} = \overset{c}{\delta}_k g_{i\bar{j}} - G^l_{ik} g_{l\bar{j}} - G^{\bar{m}}_{\bar{j}k} g_{i\bar{m}} = g_{i\bar{j}\overset{R}{|}k} + (\overset{c}{L}{}^l_{ik} - G^l_{ik}) g_{l\bar{j}} + (\overset{c}{L}{}^{\bar{m}}_{\bar{j}k} - G^{\bar{m}}_{\bar{j}k}) g_{i\bar{m}}$, where $\overset{R}{|}$ is h-covariant derivative with respect to the connection $R\Gamma$. Since $R\Gamma$ is h-metrical, then $g_{i\bar{j}\overset{B}{|}k} = (\overset{c}{L}{}^l_{ik} - G^l_{ik}) g_{l\bar{j}} - (\overset{c}{L}{}^{\bar{m}}_{\bar{j}k} - G^{\bar{m}}_{\bar{j}k}) g_{i\bar{m}}$, and thus the claim follows. □

Definition 2.2.3. *Let (M, F) be an n-dimensional complex Finsler space. (M, F) is called G-Landsberg space if it is Landsberg and the spray coefficients G^i are holomorphic with respect to η, i.e. $\dot{\partial}_{\bar{k}} G^i = 0$.*

A few immediately consequences follow below.

Proposition 2.2.4. *If (M,F) is a G-Landsberg space then the connection $B\Gamma$ is of $(1,0)$-type.*

Corollary 2.2.5. *G^i are holomorphic with respect to η if and only if the connection $B\Gamma$ is of $(1,0)$-type.*

Proposition 2.2.6. *G^i are holomorphic with respect to η if and only if the horizontal coefficients G^i_{jk} of $B\Gamma$ depend only on z.*

Proof. If G^i are holomorphic functions with respect to η, then $\dot{\partial}_{\bar{k}}G^i = 0$ which leads to $\dot{\partial}_{\bar{k}}\overset{c}{N^i_h} = 0$ and $\dot{\partial}_{\bar{k}}G^i_{jh} = 0$. Thus, the functions G^i_{jh} are holomorphic with respect to η, too.

Now, we make a similar reasoning like that in [66, Proposition 1.1], but here for the functions G^i_{jh} that are homogeneous of degree 0 with respect to η (i.e. $(\dot{\partial}_k G^i_{jh})\eta^k = 0$). We consider $D_\varepsilon = \{\eta \in T'_z M \mid F(z,\eta) < \varepsilon,\ \varepsilon > 0\}$, with ε sufficiently small, and we study the functions G^i_{jh} on $D_{\frac{1}{\varepsilon}} \setminus D_\varepsilon$. Since these functions are homogeneous of degree 0 with respect to η, their moduli achieve a maximum at an interior point of $D_{\frac{1}{\varepsilon}} \setminus D_\varepsilon$. Thus, we can apply the strong maximum principle, which gives that the functions G^i_{jh} are constant with respect to η on $D_{\frac{1}{\varepsilon}} \setminus D_\varepsilon$. Now, letting $\varepsilon \to 0$, it turns out that the functions G^i_{jh} are locally constant along of $\eta \in T'_z M \setminus \{0\}$. Under a change of the local coordinates (z^i,η^i) into (z'^i,η'^i), the functions G^i_{jh} hold the the relation $G'^i_{jk} = \frac{\partial z'^i}{\partial z^r}\frac{\partial z^s}{\partial z'^j}\frac{\partial z^q}{\partial z'^k}G^r_{sq} + \frac{\partial z'^i}{\partial z^r}\frac{\partial^2 z^r}{\partial z'^j \partial z'^k}$. It turns out that G'^i_{jk} depend only on z', too. Thus, globally we have $G^i_{jk}(z)$. Conversely, if $G^i_{jk}(z)$ then $\dot{\partial}_{\bar{k}}G^i_{jh} = 0$, which contracted by $\eta^j\eta^h$ turns out our claim. □

Corollary 2.2.7. *The coefficients G^i_{jk} depend only on z if and only if $\dot{\partial}_k G^i_{jh} = 0$.*

Proof. It is obvious the fact that if G^i_{jk} depend only on z, then $\dot{\partial}_k G^i_{jh} = 0$. Conversely, if $\dot{\partial}_k G^i_{jh} = 0$, by conjugation we have $\dot{\partial}_{\bar{k}} G^{\bar{\imath}}_{\bar{j}\bar{h}} = 0$, i.e. $G^{\bar{\imath}}_{\bar{j}\bar{h}}$ are holomorphic with respect to η. Since the functions $G^{\bar{\imath}}_{\bar{j}\bar{h}}$ are also homogeneous of degree 0, by the same arguments as in the proof of Proposition 2.2.6 one has that $\dot{\partial}_k G^{\bar{\imath}}_{\bar{j}\bar{h}} = 0$, and by conjugation $\dot{\partial}_{\bar{k}} G^i_{jh} = 0$. Applying again Proposition 2.2.6, we get $G^i_{jk}(z)$. □

Theorem 2.2.8. *Let (M,F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:*
i) (M,F) is a G-Landsberg space;
ii) $G^i_{jk} = \overset{c}{L^i_{jk}}(z)$;
iii) $C_{l\bar{r}h\,\overset{B}{|}\,0} = 0$ and $C_{j\bar{0}h\,\overset{B}{|}\,\bar{0}} = 0$;
iv) $g_{i\bar{j}\,\overset{B}{|}\,k} = \overset{c}{L^{\bar{m}}_{\bar{j}k}}\, g_{i\bar{m}}$ and $\dot{\partial}_{\bar{h}}G^i = 0$.
v) $C_{j\bar{r}h\,\overset{B}{|}\,k} + C_{k\bar{r}h\,\overset{B}{|}\,j} = 0$ and $C_{r\bar{l}h\,\overset{B}{|}\,\bar{k}} + C_{r\bar{k}h\,\overset{B}{|}\,\bar{l}} = 0$.

Proof. i) ⇔ ii) is a direct consequence of Proposition 2.2.6. i) ⇔ iii) results by Lemma 1.2.1 iii) and Theorem 2.2.2 ii). Under assumptions $\dot{\partial}_{\bar{h}}G^i = 0$, the equivalence i) ⇔ iv) from Theorem 2.2.2 provides the proof for i) ⇔ iv).

i) $\Rightarrow$ v) If (M,F) is a G-Landsberg space then by Lemma 1.2.1 ii) and v) we have

$$g_{l\bar{r}\overset{B}{|}h}+g_{h\bar{r}\overset{B}{|}l}=0 \ \text{ and } \ C_{l\bar{r}\bar{h}\overset{B}{|}k}=\dot{\partial}_{\bar{h}}(g_{l\bar{r}\overset{B}{|}k}).$$

Thus, it turns out that $C_{l\bar{r}\bar{h}\overset{B}{|}k}+C_{k\bar{r}\bar{h}\overset{B}{|}l}=\dot{\partial}_{\bar{h}}(g_{l\bar{r}\overset{B}{|}k}+g_{k\bar{r}\overset{B}{|}l})=0$ and by conjugation, the last relation leads to $C_{r\bar{l}h\overset{B}{|}\bar{k}}+C_{r\bar{k}h\overset{B}{|}\bar{l}}=0$. Now, using Lemma 1.2.1 iv) and Proposition 2.2.6, it follows that $C_{j\bar{r}h\overset{B}{|}k}+C_{k\bar{r}h\overset{B}{|}j}=0$.

v) $\Rightarrow$ i) First, contracting with η^k the identity $C_{j\bar{r}h\overset{B}{|}k}+C_{k\bar{r}h\overset{B}{|}j}=0$, it results $C_{j\bar{r}h\overset{B}{|}0}=0$, i.e. the space is Landsberg. On the other hand, the contraction by $\bar{\eta}^k\bar{\eta}^l$ of the identity $C_{r\bar{l}h\overset{B}{|}\bar{k}}+C_{r\bar{k}h\overset{B}{|}\bar{l}}=0$ gives $2C_{r\bar{0}h\overset{B}{|}\bar{0}}=0$ and by conjugation, this is $2C_{0\bar{r}\bar{h}\overset{B}{|}0}=0$. Using Lemma 1.2.1 iii), we have $2(\dot{\partial}_{\bar{h}}G^i)g_{i\bar{r}}=C_{0\bar{r}\bar{h}\overset{B}{|}0}$. From here we obtain that $\dot{\partial}_{\bar{h}}G^i=0$, which completes the proof. □

Having in mind the tensorial characterization iii) from Theorem 2.2.8 for a G-Landsberg space, this give rise to another class of complex Landsberg spaces.

Definition 2.2.9. *Let (M,F) be an n-dimensional complex Finsler space. (M,F) is called strong Landsberg space if $C_{l\bar{r}h\overset{B}{|}0}=0$ and $C_{j\bar{r}h\overset{B}{|}\bar{0}}=0$.*

Theorem 2.2.10. *Let (M,F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:*
i) (M,F) is a strong Landsberg space;
ii) $g_{l\bar{r}\overset{B}{|}s}(z)$ and $\dot{\partial}_{\bar{h}}G^i=0$;
iii) $C_{l\bar{r}h\overset{B}{|}k}=0$ and $\dot{\partial}_{\bar{h}}G^i=0$;
iv) $C_{j\bar{r}h\overset{B}{|}\bar{k}}=0$.

Proof. i) $\Rightarrow$ ii). If (M,F) is a strong Landsberg space, then by Theorem 2.2.8 iii) it is G-Landsberg. Therefore, by Lemma 1.2.1 iv) and v) one has that $C_{i\bar{j}h\overset{B}{|}k}=\dot{\partial}_h(g_{i\bar{j}\overset{B}{|}k})$ and $C_{l\bar{r}\bar{h}\overset{B}{|}k}=\dot{\partial}_{\bar{h}}(g_{l\bar{r}\overset{B}{|}k})$, which contracted by η^k lead to

$$\dot{\partial}_h(g_{i\bar{j}\overset{B}{|}k})\eta^k=\dot{\partial}_{\bar{h}}(g_{l\bar{r}\overset{B}{|}k})\eta^k=0. \tag{2.1}$$

Differentiating the second equality in (2.1) by η^s it yields

$$0=\dot{\partial}_{\bar{h}}[\dot{\partial}_s(g_{l\bar{r}\overset{B}{|}k})\eta^k]+\dot{\partial}_{\bar{h}}(g_{l\bar{r}\overset{B}{|}s}).$$

Now, using the first relation from (2.1) it results $\dot{\partial}_{\bar{h}}(g_{l\bar{r}\overset{B}{|}s})=0$. Since $g_{l\bar{r}\overset{B}{|}s}$ are holomorphic and homogeneous of degree zero with respect to η, one has that $g_{l\bar{r}\overset{B}{|}s}$ depends only on z, i.e. $g_{l\bar{r}\overset{B}{|}s}(z)$. Now, the conditions $g_{l\bar{r}\overset{B}{|}s}(z)$ and $\dot{\partial}_{\bar{h}}G^i=0$ substituted into Lemma 1.2.1 iv), give $C_{i\bar{j}h\overset{B}{|}k}=0$. Thus, we have proved ii) $\Rightarrow$ iii).

To prove iii) $\Rightarrow$ iv) we use again Lemma 1.2.1 iv). Under assumptions iii), $\dot{\partial}_h(g_{i\bar{j}\stackrel{B}{|}k}) = 0$, and by conjugation, it follows that $\dot{\partial}_{\bar{h}}(g_{j\bar{\imath}\stackrel{B}{|}\bar{k}}) = 0$. This means that the functions $g_{j\bar{\imath}\stackrel{B}{|}\bar{k}}$ are holomorphic with respect to η. Making use of their homogeneity it turns out that $g_{j\bar{\imath}\stackrel{B}{|}\bar{k}}(z)$ and thus the conjugates $g_{i\bar{j}\stackrel{B}{|}k}$ depend on z only. Therefore, v) from Lemma 1.2.1 leads to $C_{l\bar{r}\bar{h}\stackrel{B}{|}k} = 0$, this is iv).
The proof is complete if we show that iv) $\Rightarrow$ i). Indeed, $C_{l\bar{r}\bar{h}\stackrel{B}{|}k} = 0$ implies $C_{l\bar{r}\bar{h}\stackrel{B}{|}0} = 0$ and $\dot{\partial}_{\bar{h}}G^i = 0$, by Lemma 1.2.1 iii). Lemma 1.2.1 v) gives that $\dot{\partial}_{\bar{h}}(g_{i\bar{j}\stackrel{B}{|}k}) = 0$ and also $g_{j\bar{\imath}\stackrel{B}{|}\bar{k}}(z)$. Thus, by Lemma 1.2.1 iv), we obtain that $C_{l\bar{r}h\stackrel{B}{|}k} = 0$, which contracted by η^k yields $C_{l\bar{r}h\stackrel{B}{|}0} = 0$. So, the space is strong Landsberg. □

Remark 2.2.11. *By Theorem 2.2.8 iv) and Theorem 2.2.10 ii) it follows that an n-dimensional complex Finsler space is strong Landsberg if and only if $(\overset{c}{L}{}^{\bar{m}}_{\bar{j}k}g_{i\bar{m}})(z)$ and $\dot{\partial}_{\bar{h}}G^i = 0$.*

Having in the mind that any Kähler complex Finsler space is necessarly complex Landsberg, we can introduce another generalization for the G-Landsberg spaces. So, by replacing the Landsberg condition from definition of the G-Landsberg space with the Kähler condition we obtain:

Definition 2.2.12. *Let (M, F) be an n-dimensional complex Finsler space. (M, F) is called G-Kähler space if it is Kähler and the spray coefficients G^i are holomorphic with respect to η.*

A few necessary and sufficient conditions for G-Kähler spaces are given by the next theorem.

Theorem 2.2.13. *Let (M, F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:*
i) (M, F) is G-Kähler;
ii) $G^i_{j\bar{k}} = \overset{c}{L}{}^i_{j\bar{k}}$;
iii) $G^i_{jk} = L^i_{jk}(z)$;
iv) (M, F) is Kähler-Berwald space;
v) $g_{i\bar{j}\stackrel{B}{|}k} = 0$ and $\dot{\partial}_{\bar{h}}G^i = 0$.

Proof. i) $\Leftrightarrow$ ii). If (M, F) is G-Kähler then $\overset{c}{L}{}^i_{j\bar{k}} = 0$ and $G^i_{j\bar{k}} = 0$. These imply that $G^i_{j\bar{k}} = \overset{c}{L}{}^i_{j\bar{k}}$.
Conversely, if $G^i_{j\bar{k}} = \overset{c}{L}{}^i_{j\bar{k}}$ then $\dot{\partial}_{\bar{k}}\overset{c}{N}{}^i_j = \frac{1}{2}g^{\bar{l}i}(\overset{c}{\delta}_{\bar{k}}g_{j\bar{l}} - \overset{c}{\delta}_{\bar{l}}g_{j\bar{k}})$, which contracted by $\bar{\eta}^k$ gives

$$\bar{\eta}^k(\overset{c}{\delta}_{\bar{k}}g_{j\bar{l}}) - (\overset{c}{\delta}_{\bar{l}}g_{j\bar{k}})\bar{\eta}^k = 0.$$

Since $\bar{\eta}^k\overset{c}{\delta}_{\bar{k}} = \bar{\eta}^k\delta_{\bar{k}}$ and $(\overset{c}{\delta}_{\bar{l}}g_{j\bar{k}})\bar{\eta}^k = (\delta_{\bar{l}}g_{j\bar{k}})\bar{\eta}^k$, it follows that $(\delta_{\bar{k}}g_{j\bar{l}} - \delta_{\bar{l}}g_{j\bar{k}})\bar{\eta}^k = 0$ which means that (M, F) is Kähler, as well as $\dot{\partial}_{\bar{k}}\overset{c}{N}{}^i_j = 0$. The contraction of $\dot{\partial}_{\bar{k}}\overset{c}{N}{}^i_j = 0$ by η^j gives $\dot{\partial}_{\bar{k}}G^i = 0$, and thus, G^i does not depend on $\bar{\eta}^k$.

Taking into account that (M,F) is Kähler if and only if $L^i_{jk} = \overset{c}{L^i_{jk}} = G^i_{jk}$, and using Proposition 2.2.6, the claims i) $\Leftrightarrow$ iii) and i) $\Leftrightarrow$ iv) follow.

i) $\Leftrightarrow$ v). It is obvious the fact that, if (M,F) is G-Kähler, then $g_{i\bar{j}\overset{B}{|}k} = g_{i\bar{j}|k} = 0$. Conversely, if $g_{i\bar{j}\overset{B}{|}k} = 0$ and $\dot{\partial}_{\bar{h}} G^i = 0$, then $G^i_{jk} = g^{\bar{l}i}(\overset{c}{\delta}_k g_{j\bar{l}})$. Since $G^i_{jk} = G^i_{kj}$ then $\overset{c}{\delta}_k g_{j\bar{l}} = \overset{c}{\delta}_j g_{k\bar{l}}$ and $(\overset{c}{\delta}_k g_{j\bar{l}})\eta^k = (\overset{c}{\delta}_j g_{k\bar{l}})\eta^k$. The later means that the space is Kähler. □

An immediately consequence of Theorem 2.2.13 follows.

Proposition 2.2.14. *(M,F) is a Kähler-Berwald space if and only if the connections $B\Gamma$ and $R\Gamma$ are of $(1,0)$-type.*

Lemma 2.2.15. *For any complex Finsler space (M,F), $C_{l\bar{r}h|k} = 0$ if and only if $C_{l\bar{r}h|\bar{k}} = 0$.*

Proof. If $C_{l\bar{r}h|k} = 0$, then by Lemma 1.1.1 i), it follows that $\dot{\partial}_h L^i_{lk} = 0$ and the conjugates $\dot{\partial}_{\bar{h}} L^{\bar{i}}_{\bar{l}\bar{k}} = 0$. This means that $L^{\bar{i}}_{\bar{l}\bar{k}}$ are holomorphic with respect to η, which together with their homogeneity of degree 0, gives $L^i_{jk}(z)$. Thus, by Lemma 1.1.1 ii) it turns out that $C_{l\bar{r}h|\bar{k}} = 0$. Conversely, if $C_{l\bar{r}h|\bar{k}} = 0$ then the condition ii) from Lemma 1.1.1 is reduced to $(\dot{\partial}_{\bar{h}} L^i_{lk}) g_{i\bar{r}} + (\dot{\partial}_{\bar{h}} N^i_k) C_{i\bar{r}l} = 0$. By contracting the last relation with η^l, it follows that $\dot{\partial}_{\bar{h}} N^i_k = 0$. Thus, it turns out that $\dot{\partial}_{\bar{h}} L^i_{lk} = 0$. Now, using i) from Lemma 1.1.1, one has $C_{l\bar{r}h|k} = 0$. □

We note that $C_{l\bar{r}h|\bar{k}} = 0$ or $C_{l\bar{r}h|k} = 0$ implies $\dot{\partial}_{\bar{h}} G^i = 0$, but in general the converse is not true. The condition $\dot{\partial}_{\bar{h}} G^i = 0$ together with the Kähler property gives either $C_{l\bar{r}h|\bar{k}} = 0$ or $C_{l\bar{r}h|k} = 0$. Therefore, a tensorial characterizations for Kähler-Berwald spaces is provided by the next theorem.

Theorem 2.2.16. *(M,F) is Kähler-Berwald space if and only if it is Kähler and either $C_{l\bar{r}h|\bar{k}} = 0$ or $C_{l\bar{r}h|k} = 0$.*

In the remainder of this section we return to the notion of the real Berwald space, [50]. It is a real Finsler space for which the coefficients of the (real) Berwald connection depend only on the position. Our problem is to see whether there exist a corespondent of this real assertion in complex Finsler geometry. Taking into account Theorem 2.2.13 we have $G^i_{jk}(z)$, for any complex Berwald space. Nevertheless the converse is not true. As one has emphasized below, there are complex Finsler spaces with G^i_{jk} depending only on z which are not Berwald. Therefore, there comes into view another class of complex Finsler spaces.

Definition 2.2.17. *Let (M,F) be an n-dimensional complex Finsler space. (M,F) is called generalized Berwald if the horizontal coefficients G^i_{jk} of $B\Gamma$ depend only on the position z.*

Using Corollary 2.2.5 and Proposition 2.2.6, we have proved the following result.

Theorem 2.2.18. *Let (M,F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:*
i) (M,F) is generalized Berwald;
ii) G^i are holomorphic with respect to η;
iii) $B\Gamma$ is of $(1,0)$-type.

Corollary 2.2.19. *If (M,F) is a complex Berwald space, then the space is generalized Berwald.*

Proof. Since the coefficients L^i_{jk} depend only on z, we have $\dot{\partial}_{\bar{h}}L^i_{jk}=0$, which contracted by $\eta^j\eta^k$ gives $\dot{\partial}_{\bar{h}}G^i=0$. □

We note that in the particular case of the pure Hermitian metrics (i.e. $g_{i\bar{j}}=g_{i\bar{j}}(z)$), the notions of complex Berwald and generalized Berwald coincide. Summing up all the results proved the above, the inclusions from Figure 2.1 seem natural. The intersection of the sets of complex Landsberg and generalized Berwald spaces gives the class of G-Landsberg spaces.

An example of complex Berwald space is given by the complex version of *Antonelli-Shimada* metric

$$F^2_{AS}=L_{AS}(z,w;\eta,\theta)=e^{2\sigma}\left(|\eta|^4+|\theta|^4\right)^{\frac{1}{2}}\text{, with }\ \eta,\theta\neq 0, \tag{2.2}$$

on a domain D from $\widetilde{T'M}$, $\dim M=2$, such that its metric tensor is nondegenerated. We relabeled the local coordinates z^1, z^2, η^1, η^2 as z, w,η, θ, respectively. $\sigma(z,w)$ is a real valued function, [116, 37]. A direct computation leads to the following non-zero coefficients:

$$L^1_{11}=L^2_{21}=2\frac{\partial\sigma}{\partial z}\text{ and }L^1_{12}=L^2_{22}=2\frac{\partial\sigma}{\partial w},$$

which depend only on z and w. Also, we get

$$G^1=(\frac{\partial\sigma}{\partial z}\eta+\frac{\partial\sigma}{\partial w}\theta)\eta\text{ and }G^2=(\frac{\partial\sigma}{\partial z}\eta+\frac{\partial\sigma}{\partial w}\theta)\theta,$$

which do not depend on $\bar{\eta}$ and $\bar{\theta}$. Thus, L_{AS} is complex Berwald and in general, it is not Kähler. Some additional computations shows that L_{AS} is not G-Landsberg ($\overset{c}{L}{}^i_{jk}\neq G^i_{jk}$). In particular, if σ is a constant, then the metric L_{AS} is locally Minkowski.

Another example of generalized Berwald space is provided by the complex version of *Wrona* metric on a subset of $\mathbb{C}^n$, [160]

$$F(z,\eta)=\frac{|PQ|}{|OH|}=\frac{|\eta|^4}{|z|^2|\eta|^2-|<z,\eta>|^2}, \tag{2.3}$$

with $(z,\eta)\in\Omega=\{(z,\eta)\in\mathbb{C}^n\times\mathbb{C}^n\mid z\neq\lambda\eta,\ \lambda\in\mathbb{C}\}$, where P, $Q\in\mathbb{C}^n$, O is the origin of $\mathbb{C}^n$, H is the projection of O on the line PQ and, $|PQ|$ and $|OH|$ are the Euclidian lengths of the segments $[PQ]$ and $[OH]$, respectively. It follows that $G^i=0$ and also $\overset{c}{L}{}^i_{jk}\neq G^i_{jk}$ which attest that (2.3) is an example of generalized Berwald metric which is neither G-Landsberg nor complex Berwald. Moreover it satisfies $\overset{c}{L}{}^i_{jk}\eta^j=G^i_{jk}\eta^j$.

It seems that may be defined and investigated a new class of complex spaces which satisfies $\overset{c}{L}{}^i_{jk}\eta^j=G^i_{jk}\eta^j$, called for example *weak Landsberg*, being a generalization of the complex Landsberg spaces.

Trivial examples of strong Landsberg metrics are provided by the pure Hermitian metrics. Moreover, any locally Minkowski manifold is Kähler-Berwald. In the next section we came with some nice families of generalized Berwald spaces.

2.3 Generalized Berwald spaces with (α,β)-metrics

Let $\tilde{a}=a_{i\bar{j}}(z)dz^i\otimes d\bar{z}^j$ be a pure Hermitian metric and let $b=b_i(z)dz^i$ be a differential $(1,0)$-form. By these tools we have defined (for more details see [36, 27]) the complex (α,β)-metric

F on $T'M$,

$$F(z,\eta)=F(\alpha(z,\eta),|\beta(z,\eta)|), \tag{2.4}$$

where $\alpha(z,\eta)=\sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$ and $\beta(z,\eta)=b_i(z)\eta^i$. Let us recall that the coefficients of the Chern-Finsler connection corresponding to the pure Hermitian metric α are given by the formulas

$$\overset{a}{N}{}^k_j=a^{\bar{m}k}\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l,\quad \overset{a}{L}{}^i_{jk}=a^{\bar{l}i}(\overset{a}{\delta}_k a_{j\bar{l}}),\quad \overset{a}{C}{}^i_{jk}=0$$

and we consider also the settings $b^i=a^{\bar{j}i}b_{\bar{j}},\ \ ||b||^2=a^{\bar{j}i}b_ib_{\bar{j}},\ \ b^{\bar{i}}=\bar{b}^i$.

Lemma 2.3.1. *[36] Let (M,F) be a complex Finsler space with (α,β)-metric that holds $\frac{\partial|\beta|^2}{\partial z^i}=||b||^2\frac{\partial\alpha^2}{\partial z^i}$. The following statements are equivalent:*

i) $||b||^2b^r\frac{\partial a_{r\bar{m}}}{\partial z^l}\bar{\eta}^m=\bar{\beta}\bar{b}^m b^r\frac{\partial a_{r\bar{m}}}{\partial z^l}$;

ii) $||b||^2\frac{\partial b_{\bar{m}}}{\partial z^i}\bar{\eta}^m=\bar{\beta}\frac{\partial b_{\bar{m}}}{\partial z^i}\bar{b}^m$;

iii) $b_{\bar{s}}\frac{\partial b_{\bar{m}}}{\partial z^i}\bar{\eta}^m=\bar{\beta}\frac{\partial b_{\bar{s}}}{\partial z^i}$;

iv) $\bar{\beta}\left(\frac{\partial b_i}{\partial z^l}\eta^i\eta^l-2b_l\overset{a}{G}{}^l\right)+\beta\frac{\partial b_{\bar{m}}}{\partial z^l}\bar{\eta}^m\eta^l=0$*, where* $\overset{a}{G}{}^l=\frac{1}{2}\overset{a}{N}{}^l_j\eta^j$.

Proposition 2.3.2. *[36] Let (M,F) be a complex Finsler space with (α,β)-metric that holds $\frac{\partial|\beta|^2}{\partial z^i}=||b||^2\frac{\partial\alpha^2}{\partial z^i}$. If one of the equivalent conditions from Lemma 2.3.1 holds, then $N^i_j=\overset{a}{N}{}^i_j$. Moreover, if α is Kähler, then F is Kähler.*

Theorem 2.3.3. *Let (M,F) be a complex Finsler space with (α,β)-metric that holds the relation $\frac{\partial|\beta|^2}{\partial z^i}=||b||^2\frac{\partial\alpha^2}{\partial z^i}$. If one of the equivalent conditions from Lemma 2.3.1 holds, then (M,F) is generalized Berwald. Moreover, if α is Kähler then (M,F) is Kähler-Berwald.*

Proof. By Proposition 2.3.2, it follows that the coefficients are expressed as $G^i=a^{\bar{m}k}\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\eta^j$ and they are holomorphic with respect to η, i.e. the space is generalized Berwald. Assuming the Kähler property for $a_{i\bar{j}}$, it turns out that the space is Kähler-Berwald. □

In [26], we focused on two classes of complex (α,β)-metrics, namely the complex Randers metrics $F=\alpha+|\beta|$ and the complex Kropina metrics $F=\frac{\alpha^2}{|\beta|}$, $|\beta|\neq 0$. Subsequently, we present only a few results regarding generalized Berwald-Randers spaces.

Considering the complex Randers metric $F=\alpha+|\beta|$ we recall the following notations and formulas [27],

$$\frac{\partial\alpha}{\partial\eta^i}=\frac{1}{2\alpha}l_i,\quad \frac{\partial|\beta|}{\partial\eta^i}=\frac{\bar{\beta}}{2|\beta|}b_i,\quad \eta_i=\frac{\partial L}{\partial\eta^i}=\frac{F}{\alpha}l_i+\frac{F\bar{\beta}}{|\beta|}b_i, \tag{2.5}$$
$$N^i_j=\overset{a}{N}{}^i_j+\frac{1}{\gamma}\left(l_{\bar{r}}\frac{\partial\bar{b}^r}{\partial z^j}-\frac{\beta^2}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r\right)\xi^i+\frac{\beta}{2|\beta|}k^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j},$$

where $k^{\bar{r}i}=2\alpha a^{\bar{j}i}+\frac{2(\alpha||b||^2+2|\beta|)}{\gamma}\eta^i\bar{\eta}^r-\frac{2\alpha^3}{\gamma}b^i\bar{b}^r-\frac{2\alpha}{\gamma}(\bar{\beta}\eta^i\bar{b}^r+\beta b^i\bar{\eta}^r)$, $\gamma=L+\alpha^2(||b||^2-1)$, $\xi^i=\bar{\beta}\eta^i+\alpha^2b^i$. Consequently, the corresponding spray coefficients are given by

$$G^i=\overset{a}{G}{}^i+\frac{1}{2\gamma}\left(l_{\bar{r}}\frac{\partial\bar{b}^r}{\partial z^j}-\frac{\beta^2}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r\right)\xi^i\eta^j+\frac{\beta}{4|\beta|}k^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j. \tag{2.6}$$

Moreover, for the weakly Kähler complex Randers spaces we have proven.

Proposition 2.3.4. *[27] A complex Randers space (M,F) is weakly Kähler if and only if*

$$\frac{\alpha^2|\beta|}{\gamma\delta}\left[\beta\frac{\alpha||b||^2+|\beta|}{|\beta|}\frac{\partial b_{\bar m}}{\partial z^r}\bar\eta^m+\bar\beta\left(\frac{\partial b_r}{\partial z^l}-b^{\bar m}\frac{\partial a_{l\bar m}}{\partial z^r}\right)\eta^l-\alpha|\beta|b^{\bar m}\frac{\partial b_{\bar m}}{\partial z^r}\right]\eta^r C_k$$

$$-\left(\alpha\bar\beta F_{kl}+\alpha b_l\frac{\partial b_{\bar r}}{\partial z^k}\bar\eta^r+2|\beta|a_{l\bar r}\Gamma^{\bar r}_{\bar j k}\bar\eta^j\right)\eta^l+\alpha b_k\frac{\partial b_{\bar m}}{\partial z^r}\bar\eta^m\eta^r=0, \tag{2.7}$$

where $C_j=C_{j\bar h k}g^{\bar h k}=\delta\left(\frac{1}{\alpha^2}l_j-\frac{\bar\beta}{|\beta|^2}b_j\right)$ *with* $\delta=\frac{\alpha^2||b||^2-|\beta|^2}{2\gamma}-\frac{n|\beta|}{2F}$, $\Gamma^{\bar r}_{\bar j i}=\frac{1}{2}a^{\bar r k}(\frac{\partial a_{k\bar j}}{\partial z^i}-\frac{\partial a_{i\bar j}}{\partial z^k})$ *and* $F_{il}=\frac{\partial b_l}{\partial z^i}-\frac{\partial b_i}{\partial z^l}$.

Theorem 2.3.5. *Let (M,F) be a connected complex Randers space. (M,F) is a generalized Berwald space if and only if* $(\bar\beta l_{\bar r}\frac{\partial\bar b^r}{\partial z^j}+\beta\frac{\partial b_{\bar r}}{\partial z^j}\bar\eta^r)\eta^j=0$.

Proof. If (M,F) is generalized Berwald then $2G^i=G^i_{jk}(z)\ \eta^j\eta^k$, which means that G^i is quadratic in η. Thus, using (2.6) we have

$$\alpha|\beta|\{-\beta[(\alpha^2||b||^2+|\beta|^2)a^{\bar r i}+||b||^2\bar\eta^r\eta^i-\alpha^2\bar b^r b^i-\bar\beta\eta^i\bar b^r-\beta b^i\bar\eta^r]\tfrac{\partial b_{\bar r}}{\partial z^j}\eta^j$$

$$+4|\beta|^2(G^i-\overset{a}{G^i})\}+|\beta|^2[2(\alpha^2||b||^2+|\beta|^2)(G^i-\overset{a}{G^i})-2\alpha^2\beta a^{\bar r i}\tfrac{\partial b_{\bar r}}{\partial z^j}\eta^j$$

$$-(\bar\beta l_{\bar r}\tfrac{\partial\bar b^r}{\partial z^j}+\beta\tfrac{\partial b_{\bar r}}{\partial z^j}\bar\eta^r)\eta^j\eta^i-\tfrac{\alpha^2\beta}{|\beta|^2}(\bar\beta l_{\bar r}\tfrac{\partial\bar b^r}{\partial z^j}-\beta\tfrac{\partial b_{\bar r}}{\partial z^j}\bar\eta^r)\eta^j b^i]=0,$$

which contains an irrational part and another rational one. Thus, we obtain

$$\beta\left[(\alpha^2||b||^2+|\beta|^2)a^{\bar r i}+||b||^2\bar\eta^r\eta^i-\alpha^2\bar b^r b^i-\bar\beta\eta^i\bar b^r-\beta b^i\bar\eta^r\right]\tfrac{\partial b_{\bar r}}{\partial z^j}\eta^j$$
$$=4|\beta|^2(G^i-\overset{a}{G^i})\text{ and}$$

$$(\bar\beta l_{\bar r}\tfrac{\partial\bar b^r}{\partial z^j}+\beta\tfrac{\partial b_{\bar r}}{\partial z^j}\bar\eta^r)\eta^j\eta^i+\tfrac{\alpha^2\beta}{|\beta|^2}(\bar\beta l_{\bar r}\tfrac{\partial\bar b^r}{\partial z^j}-\beta\tfrac{\partial b_{\bar r}}{\partial z^j}\bar\eta^r)\eta^j b^i+2\alpha^2\beta a^{\bar r i}\tfrac{\partial b_{\bar r}}{\partial z^j}\eta^j$$
$$=2(\alpha^2||b||^2+|\beta|^2)(G^i-\overset{a}{G^i}).$$

Their contractions by b_i and l_i yield

$$\begin{aligned}&(G^i-\overset{a}{G^i})b_i=0;\\ &4|\beta|^2(\overset{a}{G^i}-G^i)l_i+2\beta\alpha^2(||b||^2\bar\eta^r-\bar\beta\bar b^r)\tfrac{\partial b_{\bar r}}{\partial z^j}\eta^j=0;\\ &\bar\beta(\alpha^2||b||^2+|\beta|^2)l_{\bar r}\tfrac{\partial\bar b^r}{\partial z^j}\eta^j-\beta(\alpha^2||b||^2-|\beta|^2)\tfrac{\partial b_{\bar r}}{\partial z^j}\bar\eta^r\eta^j+2\alpha^2|\beta|^2\bar b^r\tfrac{\partial b_{\bar r}}{\partial z^j}\eta^j=0;\\ &(\alpha^2||b||^2+|\beta|^2)(\overset{a}{G^i}-G^i)l_i+\alpha^2(\bar\beta l_{\bar r}\tfrac{\partial\bar b^r}{\partial z^j}+\beta\tfrac{\partial b_{\bar r}}{\partial z^j}\bar\eta^r)\eta^j=0.\end{aligned} \tag{2.8}$$

Adding the second and the third relations from (2.8), we obtain

$$4|\beta|^2(\overset{a}{G^i}-G^i)l_i+(\alpha^2||b||^2+|\beta|^2)(\bar\beta l_{\bar r}\frac{\partial\bar b^r}{\partial z^j}+\beta\frac{\partial b_{\bar r}}{\partial z^j}\bar\eta^r)\eta^j=0.$$

This together with the fourth equation from (2.8) implies

$$(\overset{a}{G^i} - G^i)l_i = 0 \text{ and } (\bar{\beta} l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^j} + \beta \frac{\partial b_{\bar{r}}}{\partial z^j} \bar{\eta}^r)\eta^j = 0.$$

Conversely, if $(\bar{\beta} l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^j} + \beta \frac{\partial b_{\bar{r}}}{\partial z^j} \bar{\eta}^r)\eta^j = 0$, by differentiation with respect to $\bar{\eta}^m$ we deduce that $(l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^j} b_{\bar{m}} + \beta \frac{\partial b_{\bar{m}}}{\partial z^j})\eta^j = 0$. The last two relations give

$$a^{\bar{m}i} \frac{\partial b_{\bar{m}}}{\partial z^j}\eta^j = \frac{\beta}{|\beta|^2} \frac{\partial b_{\bar{r}}}{\partial z^j} \bar{\eta}^r b^i \eta^j \text{ and } \bar{b}^m \frac{\partial b_{\bar{m}}}{\partial z^j}\eta^j = ||b||^2 \frac{\beta}{|\beta|^2} \frac{\partial b_{\bar{r}}}{\partial z^j} \bar{\eta}^r \eta^j,$$

which substituted into (2.6) imply that $G^i = \overset{a}{G^i}$ and thus, G^i are holomorphic in η, i.e. the space is generalized Berwald. □

Theorem 2.3.6. *Let (M, F) be a connected complex Randers space. (M, F) is a Kähler-Berwald space if and only if it is generalized Berwald and weakly Kähler.*

Proof. If (M, F) is Kähler-Berwald then it is obvious that the space is generalized Berwald and in particular, weakly Kähler. Now, we prove the converse. On the one hand, if the space is generalized Berwald, by Theorem 2.3.5, it turns out that $(\bar{\beta} l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^j} + \beta \frac{\partial b_{\bar{r}}}{\partial z^j} \bar{\eta}^r)\eta^j = 0$, which can be rewritten as

$$\bar{\beta} \left(\frac{\partial b_i}{\partial z^l} \eta^i \eta^l - 2 b_l \overset{a}{G^l} \right) + \beta \frac{\partial b_{\bar{m}}}{\partial z^l} \bar{\eta}^m \eta^l = 0. \tag{2.9}$$

Moreover, (2.9) implies that

$$||b||^2 \bar{\beta} \left(\frac{\partial b_i}{\partial z^l} \eta^i \eta^l - 2 b_l \overset{a}{G^l} \right) + |\beta|^2 \bar{b}^m \frac{\partial b_{\bar{m}}}{\partial z^l} \eta^l = 0. \tag{2.10}$$

On the second hand, the space is assumed to be weakly Kähler. Therefore, (2.9) and (2.10) substituted into (2.7) lead to

$$\alpha^2 \left(\bar{\beta} F_{kl} \eta^l + \beta \frac{\partial b_{\bar{r}}}{\partial z^k} \bar{\eta}^r - b_k \frac{\partial b_{\bar{r}}}{\partial z^l} \bar{\eta}^r \eta^l \right) + 2\alpha |\beta| a_{l\bar{r}} \Gamma^{\bar{r}}_{\bar{j}k} \bar{\eta}^j = 0, \tag{2.11}$$

which contains two parts: the first is rational and the second is irrational. It results that

$$\bar{\beta} F_{kl} \eta^l + \beta \frac{\partial b_{\bar{r}}}{\partial z^k} \bar{\eta}^r - b_k \frac{\partial b_{\bar{m}}}{\partial z^l} \bar{\eta}^m \eta^l = 0 \text{ and } a_{l\bar{r}} \Gamma^{\bar{r}}_{\bar{j}k} \bar{\eta}^j = 0. \tag{2.12}$$

The second condition from (2.12) gives the Kähler property for α. Thus, differentiating (2.9) with respect to η^k it follows that

$$b_k \frac{\partial b_{\bar{r}}}{\partial z^l} \bar{\eta}^r \eta^l = -\beta \frac{\partial b_{\bar{r}}}{\partial z^k} \bar{\eta}^r - \bar{\beta} \left(\frac{\partial b_l}{\partial z^k} + \frac{\partial b_k}{\partial z^l} \right) \eta^l + 2\bar{\beta} b_l \overset{a}{N^l_k}. \tag{2.13}$$

Now, (2.13) together with the first condition from (2.12) implies that

$$\bar{\beta} l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^k} + \beta \frac{\partial b_{\bar{r}}}{\partial z^k} \bar{\eta}^r = 0 \tag{2.14}$$

and from this, it follows its derivative with respect to $\bar{\eta}^m$

$$l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^k}b_{\bar{m}} + \beta\frac{\partial b_{\bar{m}}}{\partial z^k} = 0. \tag{2.15}$$

Moreover, the relations (2.14) and (2.15) imply that

$$a^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^k} = \frac{\beta}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^k}\bar{\eta}^r b^i \quad \text{and} \quad \bar{b}^m\frac{\partial b_{\bar{m}}}{\partial z^k} = ||b||^2\frac{\beta}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^k}\bar{\eta}^r. \tag{2.16}$$

Plugging (2.14) and (2.16) into (2.5), we obtain $N_j^i = \overset{a}{N}{}_j^i$ and thus, $L_{kj}^i = \overset{a}{L}{}_{kj}^i = \overset{a}{L}{}_{jk}^i = L_{jk}^i$, i.e. the Randers space (M, F) is Kähler which proves our claim. □

It is worth mentioning that the last result (Theorem 2.3.6) is valid for any complex Finsler space, [24]. Alternative proof of Theorem 2.3.6 is presented in the next chapter, for any complex Finsler space.

Chapter 3

Projectivities in complex Finsler geometry

Several subjects from projective real Finsler geometry were studied by us in complex Finsler geometry [25, 24, 23]. This chapter is focussed on the concept of projectively related complex Finsler metrics, in an attempt to approach complex variants of Rapcsák's theorem and Hilbert's fourth problem. As an application of our theory, we study the projectivities of a complex Randers metric [25].

3.1 Introduction and the main results

The problem of the projectively related real Finsler metrics is quite dated and results from the formulation of Hilbert's fourth problem: determine the metrics on an open subset in $\mathbb{R}^n$, whose geodesics are straight lines. Roughly speaking, two Finsler metrics, on a common underlying manifold, are called projectively related if they have the same geodesics as point sets. The study of the projective real Finsler spaces was initiated by L. Berwald [53, 52], his studies mainly concern two dimensional Finsler spaces. Further substantial contributions on this topic have been made by A. Rapcsák [123], R.B. Misra [108] and, especially Z. Szabo [136] and M. Matsumoto [109]. The problem of projective Finsler spaces is strongly connected to projectively related sprays, as Z. Shen pointed out in [128]. Moreover, the topic on projective real Finsler spaces continues to be of interest also for some special classes of Finsler metrics (see [46, 49, 69, 99, 54], etc.).

Based on some ideas from the real case, we introduced in complex Finsler geometry the concept of projectively related complex Finsler metrics [25]. There are meaningful differences when compared to real reasoning mainly on account that in complex Finsler geometry, the notion of a complex geodesic curve comports two nuances: one in the sense of Abate-Patrizio [1], and the other as introduced by Royden [125]. It is worth mentioning that while a complex geodesic curve in Royden's sense has been obtained under weakly Kähler condition along the curve, a complex geodesic curve in Abate-Patrizio's sense does not require any such restriction. Thus, we can state that any complex geodesic curve in Royden's sense is a complex geodesic curve in Abate-Patrizio's sense. In order to address a general characterization of the projectively related complex Finsler metrics, we have considered the complex geodesics in Abate-Patrizio's sense.

An overview of the chapter's content is below. In Section 3.2 we describe the projectively related complex Finsler metrics, pointing out the necessary and sufficient conditions for this (Theorem 3.2.6 and Corollary 3.2.7). Theorem 3.2.8 attests the invariance of the weakly Kähler property under a projective change. Complex versions of Rapcsák's theorem are given by Theorems 3.2.10, 3.2.11 and 3.2.12 and a complex Finsler solution for Hilbert's fourth problem is provided by Theorem 3.2.18. The last part of the chapter (Section 3.3) is devoted to the projectivities of the complex Randers metric $\tilde{F}=\alpha+|\beta|$. We provide the necessary and sufficient conditions for the metrics $\tilde{F}$ and α to be projectively related (Theorem 3.3.2). Based on the previous results, we prove that a complex Randers metric $\tilde{F}=\alpha+|\beta|$ defined on a domain D from $\mathbf{C}^n$ is projectively related to the complex Euclidean metric on D if and only if α is projectively related to the Euclidean metric and $\tilde{F}$ is a Kähler-Berwald metric (Theorem 3.3.3).

3.2 Projectively related complex Finsler metrics

Before presenting the concept of projectively related complex Finsler metrics, we recall the notion of the complex geodesic curve from [1]. Based on this, we can emphasize an important property of the Kähler-Berwald spaces, proved in [24].

In Abate-Patrizio's sense (see [1, p. 101]) a complex geodesic curve is provided by the equation

$$D_{T^h+\overline{T^h}}T^h=\theta^*(T^h,\overline{T^h}),$$

where T^h is the horizontal lift of the tangent vector along the curve and in the local coordinates, θ^* is expressed as

$$\theta^*=g^{\bar{m}k}g_{i\bar{p}}(L^{\bar{p}}_{\bar{j}\bar{m}}-L^{\bar{p}}_{\bar{m}\bar{j}})dz^i\wedge d\bar{z}^j\otimes\delta_k.$$

Moreover, the differential equations satisfied by a complex geodesic curve $z=z(s)$ of (M,F), with s a real parameter, can be written locally as

$$\frac{d^2z^i}{ds^2}+2G^i(z(s),\frac{dz}{ds})=\theta^{*i}(z(s),\frac{dz}{ds}),\tag{3.1}$$

where $z^i(s)$, $i=\overline{1,n}$, denote the coordinates along of the curve $z=z(s)$ and

$$2G^i=N^i_j\eta^j=\overset{c}{N^i_j}\eta^j\quad\text{and}\quad\theta^{*i}=2g^{\bar{j}i}\overset{c}{\delta_{\bar{j}}}L.$$

Since $\overset{c}{\delta_j}=\delta_j-(\overset{c}{N^k_j}-N^k_j)\dot{\partial}_k$ and $\delta_jL=0$, immediately results that $\theta^{*i}\eta_i=0$, where $\eta_i=\dot{\partial}_iL$. We note that the functions θ^{*i} are vanished if and only if the space is weakly Kähler [116]. Moreover, θ^{*i} are $(1,1)$-homogeneous with respect to η and $\bar{\eta}$ respectively, i.e. $(\dot{\partial}_k\theta^{*i})\eta^k=\theta^{*i}$ and $(\dot{\partial}_{\bar{k}}\theta^{*i})\bar{\eta}^k=\theta^{*i}$.

Lemma 3.2.1. *Let (M,F) be a complex Finsler space. Then, $(\dot{\partial}_{\bar{k}}G^i)\eta_i=0$.*

Proof. It results differentiating $G^ig_{i\bar{j}}=\frac{1}{2}\frac{\partial g_{h\bar{j}}}{\partial z^s}\eta^h\eta^s$ with respect to $\bar{\eta}^k$ and then contracting on it by $\bar{\eta}^j$. □

As a consequence of Lemma 3.2.1 it follows that the holomorphic curvature of the complex Finsler space (M,F) in direction η can be simply expressed in the terms of the spray coefficients G^i, namely

$$\mathcal{K}_F(z,\eta)=-\frac{4}{F^4}g_{k\bar{m}}\frac{\partial G^k}{\partial\bar{z}^h}\bar{\eta}^h\bar{\eta}^m.$$

Also, it is necessary to compute

$$\begin{aligned}\dot{\partial}_k\theta^{*i} &= 2\dot{\partial}_k(g^{\bar{j}i}\overset{c}{\delta}_{\bar{j}}L) = -2g^{\bar{j}l}g^{\bar{m}i}(\dot{\partial}_k g_{l\bar{m}})(\overset{c}{\delta}_{\bar{j}}L) + 2g^{\bar{j}i}\dot{\partial}_k(\overset{c}{\delta}_{\bar{j}}L)\\ &= -\theta^{*l}C^i_{kl} + 2g^{\bar{j}i}\dot{\partial}_k[\frac{\partial L}{\partial\bar{z}^j} - \overset{c}{N}{}^{\bar{r}}_{\bar{j}}(\dot{\partial}_{\bar{r}}L)]\\ &= -\theta^{*l}C^i_{kl} + 2g^{\bar{j}i}[\frac{\partial^2 L}{\partial\eta^k\partial\bar{z}^j} - (\dot{\partial}_k\overset{c}{N}{}^{\bar{r}}_{\bar{j}})(\dot{\partial}_{\bar{r}}L) - \overset{c}{N}{}^{\bar{r}}_{\bar{j}}g_{k\bar{r}}].\end{aligned}$$

Now, using Lemma 3.2.1 and $\frac{\partial^2 L}{\partial\eta^k\partial\bar{z}^j} = N^{\bar{r}}_{\bar{j}}g_{k\bar{r}}$ we obtain

$$\begin{aligned}(\dot{\partial}_k\overset{c}{N}{}^{\bar{r}}_{\bar{j}})(\dot{\partial}_{\bar{r}}L) &= (\dot{\partial}_k\overset{c}{N}{}^{\bar{r}}_{\bar{j}})\bar{\eta}_r = [\dot{\partial}_{\bar{j}}(\dot{\partial}_k G^{\bar{r}})]\bar{\eta}_r = \dot{\partial}_{\bar{j}}[(\dot{\partial}_k G^{\bar{r}})\bar{\eta}_r] - (\dot{\partial}_k G^{\bar{r}})(\dot{\partial}_{\bar{j}}\bar{\eta}_r)\\ &= -(\dot{\partial}_k G^{\bar{r}})C_{l\bar{r}\bar{j}}\eta^l,\end{aligned}$$

where $\bar{\eta}_r = \dot{\partial}_{\bar{r}}L$ and $C_{l\bar{r}\bar{j}}\eta^l = \dot{\partial}_{\bar{j}}\bar{\eta}_r$. Therefore,

$$\dot{\partial}_k\theta^{*i} = -\theta^{*l}C^i_{kl} + 2g^{\bar{j}i}[(N^{\bar{r}}_{\bar{j}} - \overset{c}{N}{}^{\bar{r}}_{\bar{j}})g_{k\bar{r}} + (\dot{\partial}_k G^{\bar{r}})C_{l\bar{r}\bar{j}}\eta^l]. \tag{3.2}$$

Theorem 3.2.2. *[24] Let (M, F) be a complex Finsler space that satisfies the weakly Kähler and generalized Berwald conditions. Then (M, F) is a Kähler-Berwald space.*

Proof. Under given assumptions, the relation (3.2) is $2g^{\bar{j}i}(N^{\bar{r}}_{\bar{j}} - \overset{c}{N}{}^{\bar{r}}_{\bar{j}})g_{k\bar{r}} = 0$, which contracted by $\frac{1}{2}g_{i\bar{m}}g^{\bar{s}k}$ gives $N^{\bar{s}}_{\bar{m}} - \overset{c}{N}{}^{\bar{s}}_{\bar{m}} = 0$, i.e. F is Kähler. This, together with the assumption that F is generalized Berwald, proves our claim. □

Let us consider $\tilde{L}$ another complex Finsler metric on the complex manifold M.

Definition 3.2.3. *The complex Finsler metrics L and $\tilde{L}$ on the complex manifold M, are called projectively related if they have the same complex geodesics as point sets.*

This means that for any complex geodesic $z = z(s)$ of (M, L) there is a transformation of the parameter s, such that $\tilde{s} = \tilde{s}(s)$, with $\frac{d\tilde{s}}{ds} > 0$ and $z = z(\tilde{s})$ is a geodesic of $(M, \tilde{L})$ and, conversely.

We assume that $z = z(s)$ is a complex geodesic of (M, L). Thus, it satisfies (3.1). Taking an arbitrary transformation of the parameter $t = t(s)$, with $\frac{dt}{ds} > 0$, generally, the equations (3.1) cannot be preserved. Indeed, for the new parameter t we have

$$\frac{dz^i}{ds} = \frac{dz^i}{dt}\frac{dt}{ds}, \quad \frac{d^2z^i}{ds^2} = \frac{d^2z^i}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{dz^i}{dt}\frac{d^2t}{ds^2}, \quad \theta^{*k}\left(z, \frac{dz}{ds}\right) = \left(\frac{dt}{ds}\right)^2\theta^{*k}(z, \frac{dz}{dt}).$$

Then,

$$\begin{aligned}\left[\frac{d^2z^i}{dt^2} + 2G^i(z, \frac{dz}{dt}) - \theta^{*i}(z, \frac{dz}{dt})\right]\left(\frac{dt}{ds}\right)^2 &= \frac{d^2z^i}{ds^2} - \frac{dz^i}{dt}\frac{d^2t}{ds^2} + 2G^i(z, \frac{dz}{ds}) - \theta^{*i}(z, \frac{dz}{ds})\\ &= -\frac{dz^i}{dt}\frac{d^2t}{ds^2}.\end{aligned}$$

Therefore, we get the equations (3.1) in parameter t,

$$\frac{d^2z^i}{dt^2}+2G^i(z(t),\frac{dz}{dt})=\theta^{*i}(z(t),\frac{dz}{dt})-\frac{dz^i}{dt}\frac{d^2t}{ds^2}\frac{1}{\left(\frac{dt}{ds}\right)^2},\quad i=\overline{1,n}, \tag{3.3}$$

which is equivalent to

$$\frac{\frac{d^2z^i}{dt^2}+2G^i(z,\frac{dz}{dt})-\theta^{*i}(z,\frac{dz}{dt})}{\frac{dz^i}{dt}}=-\frac{d^2t}{ds^2}\frac{1}{\left(\frac{dt}{ds}\right)^2},\quad i=\overline{1,n}. \tag{3.4}$$

Corresponding to the complex Finsler metric $\tilde{L}$ on the same manifold M, we have the spray coefficients $\tilde{G}^i$ and the functions $\tilde{\theta}^{*i}$. If L and $\tilde{L}$ are projectively related, then $z=z(\tilde{s}(s))$ is a complex geodesic of $(M,\tilde{L})$, where $\tilde{s}$ is the parameter corresponding to $\tilde{L}$. Now, we assume that the same parameter t is transformed by $t=t(\tilde{s})$ and as the above we obtain

$$\frac{\frac{d^2z^i}{dt^2}+2\tilde{G}^i(z,\frac{dz}{dt})-\tilde{\theta}^{*i}(z,\frac{dz}{dt})}{\frac{dz^i}{dt}}=-\frac{d^2t}{d\tilde{s}^2}\frac{1}{\left(\frac{dt}{d\tilde{s}}\right)^2},\quad i=\overline{1,n}. \tag{3.5}$$

The difference between (3.4) and (3.5) gives

$$2\tilde{G}^i(z,\frac{dz}{dt})-\tilde{\theta}^{*i}(z,\frac{dz}{dt})=2G^i(z,\frac{dz}{dt})-\theta^{*i}(z,\frac{dz}{dt})+\left[\frac{d^2t}{ds^2}\frac{1}{\left(\frac{dt}{ds}\right)^2}-\frac{d^2t}{d\tilde{s}^2}\frac{1}{\left(\frac{dt}{d\tilde{s}}\right)^2}\right]\frac{dz^i}{dt}, \tag{3.6}$$

$i=\overline{1,n}$. Since the equations (3.6) is satisfied along any geodesic curve, they can be rewritten as

$$2\tilde{G}^i(z,\eta)-2\tilde{\theta}^{*i}(z,\eta)=G^i(z,\eta)-\theta^{*i}(z,\eta)+B^i(z,\eta)+2P(z,\eta)\eta^i,\quad i=\overline{1,n} \tag{3.7}$$

$$2\tilde{G}^i(z,\frac{dz}{dt})-\tilde{\theta}^{*i}(z,\frac{dz}{dt})=2G^i(z,\frac{dz}{dt})-\theta^{*i}(z,\frac{dz}{dt})+2P(z,\frac{dz}{dt})\frac{dz^i}{dt},\quad i=\overline{1,n} \tag{3.8}$$

for some smooth functions P on $T'M$. Using the notation $B^i=\frac{1}{2}(\tilde{\theta}^{*i}-\theta^{*i})$, the homogeneity of the functions $\tilde{\theta}^{*i}$ and θ^{*i} give $(\dot{\partial}_kB^i)\eta^k=B^i$ and $(\dot{\partial}_{\bar{k}}B^i)\bar{\eta}^k=B^i$. Based on these, the relations (3.7) are simplified as

$$\tilde{G}^i=G^i+B^i+P\eta^i. \tag{3.9}$$

Now we are going to point out some properties for P. To do this, we redeem the homogeneities of the functions which are included in (3.9), going from η to $\lambda\eta$ for any $\lambda\in\mathbb{C}$, $\lambda\neq0$. More precisely, differentiating in (3.9) with respect to η and $\bar{\eta}$ and then setting $\lambda=1$, it turns out

$$B^i=[(\dot{\partial}_kP)\eta^k-P]\eta^i\quad\text{and}\quad B^i=-(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\eta^i \tag{3.10}$$

and thus, for every $i=\overline{1,n}$,

$$(\dot{\partial}_kP)\eta^k+(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k=P. \tag{3.11}$$

Lemma 3.2.4. *Between the spray coefficients $\tilde{G}^i$ and G^i of the metrics L and $\tilde{L}$ on M there are the relations (3.9), where P is a smooth function on $T'M$, if and only if $\tilde{G}^i=G^i+(\dot{\partial}_kP)\eta^k\eta^i$, $B^i(z,\eta)=-(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\eta^i$ and $(\dot{\partial}_kP)\eta^k+(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k=P$, for any $i=\overline{1,n}$.*

From the above considerations, we get

Lemma 3.2.5. *If the complex Finsler metrics L and $\tilde{L}$ are projectively related, then there is a smooth function P on $T'M$ satisfying $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$, such that*

$$\tilde{G}^i(z,\eta) = G^i(z,\eta) + (\dot{\partial}_k P)\eta^k\eta^i \text{ and } B^i(z,\eta) = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k\eta^i, \quad i = \overline{1,n}. \tag{3.12}$$

If we consider the notations $V = (\dot{\partial}_k P)\eta^k$ and $Q = -2(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k$, it results that $P = V - \frac{1}{2}Q$. Moreover, taking into account again the homogeneity of the functions $\tilde{G}^i$, G^i and B^i it turns out that V is $(1,0)$-homogeneous and Q is $(0,1)$-homogeneous.

Further on, we focus on the converse implication. Namely, under assumption that $z = z(s)$ is a complex geodesic of (M, L), we show that the complex Finsler metric $\tilde{L}$ with the spray coefficients $\tilde{G}^i$ given by

$$\tilde{G}^i = G^i + B^i + P\eta^i,$$

where P is a smooth function on $T'M$, is projectively related to L, i.e. there is a parametrization $\tilde{s} = \tilde{s}(s)$, with $\frac{d\tilde{s}}{ds} > 0$, such that $z = z(\tilde{s}(s))$ is a geodesic of $(M, \tilde{L})$.

If there is a parametrization $\tilde{s} = \tilde{s}(s)$ and $z = z(s)$ is a complex geodesic of (M, L), it follows that

$$\frac{d^2 z^i}{d\tilde{s}^2} = -2G^i(z, \frac{dz}{d\tilde{s}}) + \theta^{*i}(z, \frac{dz}{d\tilde{s}}) - \frac{d^2\tilde{s}}{ds^2}\frac{1}{\left(\frac{d\tilde{s}}{ds}\right)^2}\frac{dz^i}{d\tilde{s}},$$

for any $i = \overline{1,n}$, and moreover using (3.12), it leads to

$$\frac{d^2 z^i}{d\tilde{s}^2} = -2\tilde{G}^i(z, \frac{dz}{dt}) + \tilde{\theta}^{*i}(z, \frac{dz}{d\tilde{s}}) + \left(2P(z, \frac{dz}{d\tilde{s}}) - \frac{d^2\tilde{s}}{ds^2}\frac{1}{\left(\frac{d\tilde{s}}{ds}\right)^2}\right)\frac{dz^i}{d\tilde{s}}, \ i = \overline{1,n}.$$

Thus, $z = z(\tilde{s}(s))$ is a geodesic of $(M, \tilde{L})$ if and only if

$$\left(2P(z, \frac{dz}{d\tilde{s}}) - \frac{d^2\tilde{s}}{ds^2}\frac{1}{\left(\frac{d\tilde{s}}{ds}\right)^2}\right)\frac{dz^i}{d\tilde{s}} = 0, \ i = \overline{1,n}. \tag{3.13}$$

Assuming that the complex geodesic curve is not a line, it results

$$2P(z, \frac{dz}{ds})\frac{d\tilde{s}}{ds} = \frac{d^2\tilde{s}}{ds^2}. \tag{3.14}$$

Denoting by $u(s) = \frac{d\tilde{s}}{ds}$, we get $\frac{d^2\tilde{s}}{ds^2} = \frac{du}{ds}$ and so, $2P(z, \frac{dz}{ds})u = \frac{du}{ds}$. We obtain $u = ae^{\int 2P(z,\frac{dz}{ds})ds}$. From here, it results that there is $\tilde{s}(s) = a\int e^{\int 2P(z,\frac{dz}{ds})ds}ds + b$, where a, b are arbitrary constants.

Summing up all the above results, we have proven the following theorem.

Theorem 3.2.6. *Let L and $\tilde{L}$ be complex Finsler metrics on M. Then L and $\tilde{L}$ are projectively related if and only if there is a smooth function P on $T'M$, such that*

$$\tilde{G}^i = G^i + B^i + P\eta^i, \ i = \overline{1,n}. \tag{3.15}$$

As a consequence of Lemma 3.2.4 we have the following result.

Corollary 3.2.7. *Let L and $\tilde{L}$ be complex Finsler metrics on M. L and $\tilde{L}$ are projectively related if and only if there is a smooth function P on $T'M$, such that $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k\eta^i$, $B^i = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k\eta^i$ and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$, for any $i = \overline{1,n}$.*

The relations (3.15) which link the spray coefficients $\tilde{G}^i$ and G^i of the projectively related complex Finsler metrics L and $\tilde{L}$ is called *projective change.*

Theorem 3.2.8. *Let L and $\tilde{L}$ be two projectively related complex Finsler metrics on M. Then, L is weakly Kähler if and only if $\tilde{L}$ is also weakly Kähler. In this case, the projective change is $\tilde{G}^i = G^i + P\eta^i$, where P is a $(1,0)$-homogeneous function.*

Proof. Since L and $\tilde{L}$ are projectively related, then $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k\eta^i$, $B^i = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k\eta^i$ and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$. First, if L is weakly Kähler, then $\theta^{*i} = 0$. This implies that $\tilde{\theta}^{*i} = -2(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k\eta^i$, which contracted by $\tilde{g}_{i\bar{r}}\bar{\eta}^r = \dot{\partial}_i\tilde{L}$, gives $\tilde{\theta}^{*i}\tilde{g}_{i\bar{r}}\bar{\eta}^r = -2(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k\tilde{L}$. Next, $\tilde{\theta}^{*i}\tilde{g}_{i\bar{r}}\bar{\eta}^r = 0$. Thus, we get $(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = 0$, which implies $\tilde{\theta}^{*i} = 0$, i.e. $\tilde{L}$ is weakly Kähler and $P = (\dot{\partial}_k P)\eta^k$. Moreover, it follows that $\tilde{G}^i = G^i + P\eta^i$. The converse implication results immediately following the same arguments. □

Lemma 3.2.9. *Let L and $\tilde{L}$ be complex Finsler metrics on M. The spray coefficients $\tilde{G}^i$ and G^i of the metrics L and $\tilde{L}$ satisfy*

$$\tilde{G}^i = G^i + \frac{1}{2}\tilde{g}^{\bar{r}i}\left[\dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{L})\right],\ i = \overline{1,n}. \tag{3.16}$$

Proof. Starting with $\overset{c}{\delta}_k\tilde{L} = \frac{\partial\tilde{L}}{\partial z^k} - \overset{c}{N}{}^l_k(\dot{\partial}_l\tilde{L})$, by a direct computation we obtain

$$\dot{\partial}_{\bar{r}}(\overset{c}{\delta}_k\tilde{L}) = \dot{\partial}_{\bar{r}}\left(\frac{\partial\tilde{L}}{\partial z^k} - \overset{c}{N}{}^l_k(\dot{\partial}_l\tilde{L})\right) = \frac{\partial^2\tilde{L}}{\partial z^k\partial\bar{\eta}^r} - (\dot{\partial}_{\bar{r}}\overset{c}{N}{}^l_k)(\dot{\partial}_l\tilde{L}) - \overset{c}{N}{}^l_k\tilde{g}_{l\bar{r}}.$$

If we contract last relation with $\tilde{g}^{\bar{r}i}\eta^k$, and we take into account the relation $\eta^k\overset{c}{\delta}_k = \eta^k\delta_k$, it turns out that

$$\tilde{g}^{\bar{r}i}\dot{\partial}_{\bar{r}}(\overset{c}{\delta}_k\tilde{L})\eta^k = \tilde{g}^{\bar{r}i}\dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k = 2\tilde{G}^i - 2\tilde{g}^{\bar{r}i}(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{L}) - 2G^i$$

and thus, (3.16) is justified. □

Next, we prove few complex versions of the Rapcsák's theorem.

Theorem 3.2.10. *Let L and $\tilde{L}$ be complex Finsler metrics on M. Then, L and $\tilde{L}$ are projectively related if and only if*

$$\frac{1}{2}\left[\dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{L})\right] = P(\dot{\partial}_{\bar{r}}\tilde{L}) + B^i\tilde{g}_{i\bar{r}},\ r = \overline{1,n}, \tag{3.17}$$

with $P = \frac{1}{2\tilde{L}}[(\delta_k\tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{L})]$.

Proof. We assume that L and $\tilde{L}$ are projectively related. Then, by Theorem 3.2.6 and the relation (3.16) we have

$$B^i + P\eta^i = \frac{1}{2}\tilde{g}^{\bar{r}i}\left[\dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{L})\right],\ i = \overline{1,n}. \tag{3.18}$$

First, if these relations are contracted by $\tilde{g}_{i\bar{m}}\bar{\eta}^m$, we get

$$-\frac{1}{2}\theta^{*i}(\dot{\partial}_i\tilde{L}) + P\tilde{L} = \frac{1}{2}\dot{\partial}_{\bar{m}}(\delta_k\tilde{L})\eta^k\bar{\eta}^m + (\dot{\partial}_{\bar{m}}G^l)\bar{\eta}^m(\dot{\partial}_l\tilde{L}),$$

because of $B^i\tilde{g}_{i\bar{m}}\bar{\eta}^m = -\frac{1}{2}\theta^{*i}(\dot{\partial}_i\tilde{L})$.

Using the homogeneity of the functions G^i (this is $(\dot{\partial}_{\bar{m}}G^l)\bar{\eta}^m = 0$) and the fact that $\dot{\partial}_{\bar{m}}(\delta_k\tilde{L})\eta^k\bar{\eta}^m = (\delta_k\tilde{L})\eta^k$, it turns out that $P = \frac{1}{2\tilde{L}}[(\delta_k\tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{L})]$. Second, contracting the relation (3.18) only by $\tilde{g}_{i\bar{m}}$, we obtain (3.17).

Conversely, substituting the formulas (3.17) into (3.16), it leads to the relation (3.15) with $P = \frac{1}{2\tilde{L}}[(\delta_k\tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{L})]$, i.e. L and $\tilde{L}$ are projectively related. □

Theorem 3.2.11. *Let L and $\tilde{L}$ be complex Finsler metrics on M. Then, L and $\tilde{L}$ are projectively related if and only if*

$$\dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{L}) = \frac{1}{\tilde{L}}(\delta_k\tilde{L})\eta^k(\dot{\partial}_{\bar{r}}\tilde{L}), \tag{3.19}$$
$$B^r = -\frac{1}{2\tilde{L}}\theta^{*l}(\dot{\partial}_l\tilde{L})\eta^r,\ r = \overline{1,n},$$
$$P = \frac{1}{2\tilde{L}}[(\delta_k\tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{L})].$$

Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{2\tilde{L}}(\delta_k\tilde{L})\eta^k\eta^i$.

Proof. By Corollary 3.2.7, if L and $\tilde{L}$ are projectively related, then there is a smooth function P on $T'M$, such that $\tilde{G}^i = G^i + (\dot{\partial}_kP)\eta^k\eta^i$, $B^i = -(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\eta^i$ and $(\dot{\partial}_kP)\eta^k + (\dot{\partial}_{\bar{k}}P)\bar{\eta}^k = P$, for any $i = \overline{1,n}$. Using (3.16), it follows that

$$(\dot{\partial}_kP)\eta^k\eta^i = \frac{1}{2}\tilde{g}^{\bar{r}i}\left[\dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{L})\right],\ i = \overline{1,n}, \tag{3.20}$$

which contracted firstly by $\tilde{g}_{i\bar{m}}$ and secondly by $\tilde{g}_{i\bar{m}}\bar{\eta}^m$ give

$$\dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{L}) = 2(\dot{\partial}_kP)\eta^k(\dot{\partial}_{\bar{r}}\tilde{L})$$

and $(\dot{\partial}_kP)\eta^k = \frac{1}{2\tilde{L}}(\delta_k\tilde{L})\eta^k$ respectively, since $\delta_k\tilde{L}$ is $(1,1)$-homogeneous. Now, contracting $B^i = -(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\eta^i$ with $\tilde{g}_{i\bar{m}}\bar{\eta}^m$ and using the fact that $B^i\ \tilde{g}_{i\bar{m}}\bar{\eta}^m = -\frac{1}{2}\theta^{*i}(\dot{\partial}_i\tilde{L})$, it turns out that $(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k = \frac{1}{2\tilde{L}}\theta^{*i}(\dot{\partial}_i\tilde{L})$. Therefore, it follows that $P = \frac{1}{2\tilde{L}}[(\delta_k\tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{L})]$.

Conversely, substituting the first condition (3.19) into (3.16), we obtain $\tilde{G}^i = G^i + V\eta^i$, where $V = \frac{1}{2\tilde{L}}(\delta_k\tilde{L})\eta^k$. Now, since $P = \frac{1}{2\tilde{L}}[(\delta_k\tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{L})]$, we get

$$(\dot{\partial}_kP)\eta^k = \frac{1}{2\tilde{L}}(\delta_k\tilde{L})\eta^k = S \quad \text{and } (\dot{\partial}_{\bar{k}}P)\bar{\eta}^k = \frac{1}{2\tilde{L}}\theta^{*i}(\dot{\partial}_i\tilde{L}).$$

Thus, these lead to $\tilde{G}^i = G^i + (\dot{\partial}_kP)\eta^k\eta^i$, $B^i = -(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\eta^i$ and $(\dot{\partial}_kP)\eta^k + (\dot{\partial}_{\bar{k}}P)\bar{\eta}^k = P$. □

Plugging $\tilde{L} = \tilde{F}^2$ into (3.19) we have proven another equivalent complex version of Rapcsák's theorem.

Theorem 3.2.12. *Let F and $\tilde{F}$ be the complex Finsler metrics on M. Then, F and $\tilde{F}$ are projectively related if and only if*

$$\begin{aligned} \dot{\partial}_{\bar{r}}(\delta_k\tilde{F})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{F}) &= \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k(\dot{\partial}_{\bar{r}}\tilde{F}), \\ B^r &= -\frac{1}{\tilde{F}}\theta^{*l}(\dot{\partial}_l\tilde{F})\eta^r,\ r=\overline{1,n}, \\ P &= \frac{1}{\tilde{F}}[(\delta_k\tilde{F})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{F})]. \end{aligned} \tag{3.21}$$

Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k\eta^i$.

Theorem 3.2.13. *Let L be a weakly Kähler complex Finsler metric and $\tilde{L}$ another complex Finsler metric, both on M. Then, L and $\tilde{L}$ are projectively related if and only if $\tilde{L}$ is weakly Kähler and*

$$\begin{aligned} \dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{L}) &= 2P(\dot{\partial}_{\bar{r}}\tilde{L}),\ r=\overline{1,n}, \\ P &= \frac{1}{2\tilde{L}}(\delta_k\tilde{L})\eta^k. \end{aligned} \tag{3.22}$$

Moreover, the projective change is $\tilde{G}^i = G^i + P\eta^i$ and P is $(1,0)$-homogeneous.

Proof. Having in mind the Theorems 3.2.8 and 3.2.11 the direct implication is obvious. For the converse, we have $B^i = \theta^{*i} = \tilde{\theta}^{*i} = 0$, because L and $\tilde{L}$ are weakly Kähler, which together with (3.22) are sufficient conditions for the projectivity of the metrics L and $\tilde{L}$. Now, plugging (3.22) into (3.16) it follows that $\tilde{G}^i = G^i + P\eta^i$ and P is $(1,0)$-homogeneous. □

Paying more attention to Theorem 3.2.12, we obtain the following result.

Corollary 3.2.14. *Let F be a generalized Berwald metric and $\tilde{F}$ another complex Finsler metric, both on M. Then, F and $\tilde{F}$ are projectively related if and only if*

$$\begin{aligned} \dot{\partial}_{\bar{r}}(\delta_k\tilde{F})\eta^k &= \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k(\dot{\partial}_{\bar{r}}\tilde{F})\ ;\ B^r = -\frac{1}{\tilde{F}}\theta^{*l}(\dot{\partial}_l\tilde{F})\eta^r, \\ P &= \frac{1}{\tilde{F}}[(\delta_k\tilde{F})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{F})], \end{aligned} \tag{3.23}$$

for any $r=\overline{1,n}$. Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k\eta^i$ and $\tilde{F}$ is also generalized Berwald.

Proof. The equivalence results by Theorem 3.2.12, where $\dot{\partial}_{\bar{r}}G^l = 0$, because F is a generalized Berwald metric. In order to show that $\tilde{F}$ is generalized Berwald, we compute

$$\dot{\partial}_{\bar{r}}[\frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k] = -\frac{1}{\tilde{F}^2}(\dot{\partial}_{\bar{r}}\tilde{F})(\delta_k\tilde{F})\eta^k + \frac{1}{\tilde{F}}\dot{\partial}_{\bar{r}}(\delta_k\tilde{F})\eta^k = 0,$$

where we used the first identity from (3.23). Now, by differentiating the projective change $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k\eta^i$ with respect to $\bar{\eta}^r$ it follows that $\dot{\partial}_{\bar{r}}\tilde{G}^l = 0$, i.e. $\tilde{F}$ is generalized Berwald. □

In particular, if F is a Kähler-Berwald metric, then by Theorems 3.2.2, 3.2.13 and Corollary 3.2.14, we obtain the following result.

Corollary 3.2.15. *Let F be a Kähler-Berwald metric and $\tilde{F}$ another complex Finsler metric, both on M. Then, F and $\tilde{F}$ are projectively related if and only if $\tilde{F}$ is weakly Kähler and*

$$\dot{\partial}_{\bar{r}}(\delta_k\tilde{F})\eta^k = P(\dot{\partial}_{\bar{r}}\tilde{F}),\ r=\overline{1,n}\ \ and\ \ P=\frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k. \tag{3.24}$$

Moreover, the projective change is $\tilde{G}^i = G^i + P\eta^i$ and $\tilde{F}$ is Kähler-Berwald.

Proposition 3.2.16. *Let F and $\tilde{F}$ be two projectively related complex Finsler metrics on M. If P is $(1,0)$-homogeneous and F is generalized Berwald, then P is holomorphic with respect to η.*

Proof. Since $\tilde{G}^i = G^i + B^i + P\eta^i$ and P is $(1,0)$-homogenous, then $B^i = 0$. Thus, by Corollary 3.2.14, it turns out that $\theta^{*l}(\dot{\partial}_l\tilde{F}) = 0$, $P = \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k$ and, moreover $\dot{\partial}_{\bar{r}}P = 0$. □

Proposition 3.2.17. *Let F and $\tilde{F}$ be two projectively related complex Finsler metrics on M. If P is $(0,1)$-homogeneous then $B^i = -P\eta^i$ and the projective change is $\tilde{G}^i = G^i$, for any $i=\overline{1,n}$.*

Proof. Under the assumptions $\tilde{G}^i = G^i + B^i + P\eta^i$ and P is $(0,1)$-homogenous and taking into account that $\tilde{G}^i$ and G^i are $(2,0)$-homogeneous and B^i and $P\eta^i$ are $(1,1)$-homogeneous, it follows that $\tilde{G}^i = G^i$ and $B^i = -P\eta^i$. □

Further on, a complex version of the *Hilbert's fourth problem* is approached.

Theorem 3.2.18. *Let L be complex Euclidean metric on a domain D from $\mathbf{C}^n$ and $\tilde{L}$ another complex Finsler metric on D. Then, L and $\tilde{L}$ are projectively related if and only if $\tilde{L}$ is weakly Kähler and*

$$\tilde{G}^i = \frac{1}{2\tilde{L}}\frac{\partial\tilde{L}}{\partial z^k}\eta^k\eta^i,\ i=\overline{1,n}. \tag{3.25}$$

Moreover, $\tilde{L}$ is Kähler-Berwald.

Proof. The complex Euclidean metric $L = |\eta|^2 = \sum_{k=1}^n \eta^k\bar{\eta}^k$ is Kähler with the local spray coefficients $G^i = 0$, for any $i=\overline{1,n}$. By these assumptions, the conditions (3.22) can be rewritten as

$$\dot{\partial}_{\bar{r}}(\frac{\partial\tilde{L}}{\partial z^k})\eta^k = 2P(\dot{\partial}_{\bar{r}}\tilde{L}), \tag{3.26}$$

for any $r=\overline{1,n}$, where $P = \frac{1}{2\tilde{L}}\frac{\partial\tilde{L}}{\partial z^k}\eta^k$. Next, by contraction in (3.26) with $\tilde{g}^{\bar{r}i}$ and using again (3.26), it follows that $\tilde{G}^i = P\eta^i$, since $\tilde{G}^i = \frac{1}{2}\tilde{g}^{\bar{r}i}\dot{\partial}_{\bar{r}}(\frac{\partial\tilde{L}}{\partial z^k})\eta^k$. The converse is obvious. □

We note that if we replace $\tilde{L} = \tilde{F}^2$ into (3.25), it can be rewritten as $\tilde{G}^i = \frac{1}{\tilde{F}}\frac{\partial\tilde{F}}{\partial z^k}\eta^k\eta^i$, $i=\overline{1,n}$.

Example. Some examples of complex Finsler metrics that are projectively related to the complex Euclidean metric are given by the following pure Hermitian metrics

$$\tilde{F}^2(z,\eta) = \frac{|\eta|^2 + \varepsilon\left(|z|^2|\eta|^2 - |< z,\eta >|^2\right)}{(1+\varepsilon|z|^2)^2},\ \varepsilon<0, \tag{3.27}$$

defined on the disk $\Delta_r^n = \left\{z \in \mathbf{C}^n,\ |z| < r,\ \ r = \sqrt{1/|\varepsilon|}\right\} \subset \mathbf{C}^n$, where $|z|^2 = \sum_{k=1}^n z^k \bar{z}^k$, $< z, \eta >= \sum_{k=1}^n z^k \bar{\eta}^k$ and $|< z, \eta >|^2 =< z, \eta > \overline{< z, \eta >}$. We note that a direct computation leads to $\tilde{G}^i = -\frac{\varepsilon < z,\eta >}{(1+\varepsilon|z|^2)} = \frac{1}{\tilde{F}}\frac{\partial \tilde{F}}{\partial z^k}\eta^k\eta^i$. Moreover, the metrics (3.27) are Kähler with constant holomorphic curvature $\mathcal{K}_F = 4\varepsilon$. In particular, for $\varepsilon = -1$, (3.27) provides the *Bergman metric* on the unit disk $\Delta^n = \Delta_1^n$.

3.3 Projectivities of a complex Randers metric

Let us consider the complex Randers metric $\tilde{F} = \alpha + |\beta|$ on $T'M$ with $\beta(z, \eta) = b_i(z)\eta^i$ a differential $(1, 0)$-form and $\alpha(z, \eta) = \sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$ a pure Hermitian metric on M. By these objects we have defined

$$\frac{\partial \alpha}{\partial \eta^i} = \frac{1}{2\alpha} l_i, \quad \frac{\partial |\beta|}{\partial \eta^i} = \frac{\bar{\beta}}{2|\beta|} b_i, \quad \tilde{\eta}_i = \frac{\partial \tilde{L}}{\partial \eta^i} = \frac{\tilde{F}}{\alpha} l_i + \frac{\tilde{F}\bar{\beta}}{|\beta|} b_i,$$
$$\tilde{G}^i = \overset{a}{G^i} + \frac{1}{2\gamma}\left(l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j} - \frac{\beta^2}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r\right)\xi^i\eta^j + \frac{\beta}{4|\beta|}k^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j$$
$$l_i = a_{i\bar{j}}\bar{\eta}^j, \quad b^i = a^{\bar{j}i} b_{\bar{j}}, \quad ||b||^2 = a^{\bar{j}i} b_i b_{\bar{j}}, \quad b^{\bar{i}} = \bar{b}^i,$$

where $\overset{a}{G^i} = \frac{1}{2}\overset{a}{N^i_j}\eta^j$ are the spray coefficients of α, $\gamma = \tilde{L} + \alpha^2(||b||^2 - 1)$, $\xi^i = \bar{\beta}\eta^i + \alpha^2 b^i$ and $k^{\bar{r}i} = 2\alpha a^{\bar{j}i} + \frac{2(\alpha||b||^2+2|\beta|)}{\gamma}\eta^i\bar{\eta}^r - \frac{2\alpha^3}{\gamma}b^i\bar{b}^r - \frac{2\alpha}{\gamma}(\bar{\beta}\eta^i\bar{b}^r + \beta b^i\bar{\eta}^r)$.

First our aim is to determine the necessary and sufficient conditions such that the complex Randers metric $\tilde{F}$ is projectively related to the Hermitian metric α. A simple computation shows that,

$$(\delta_k\tilde{F})\eta^k = (\delta_k|\beta|)\eta^k = \frac{1}{2|\beta|}(\bar{\beta}l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^k} + \beta\frac{\partial b_{\bar{r}}}{\partial z^k}\bar{\eta}^r)\eta^k, \tag{3.28}$$

because $(\delta_k\alpha)\eta^k = 0$ and

$$\theta^{*i}(\dot{\partial}_i\tilde{F}) = -\frac{\bar{\beta}}{2|\beta|}\Gamma^k_{i\bar{j}} b_k\eta^i\bar{\eta}^j, \tag{3.29}$$

where $\delta_k = \frac{\partial}{\partial z^k} - \overset{a}{N^j_i}\dot{\partial}_j$ and $\Gamma^k_{i\bar{j}} = \frac{1}{2}a^{\bar{r}k}(\frac{\partial a_{i\bar{r}}}{\partial \bar{z}^j} - \frac{\partial a_{i\bar{j}}}{\partial \bar{z}^r})$. Taking into account Theorem 2.3.5 we have proven the following result.

Lemma 3.3.1. *Let $(M, \tilde{F})$ be a connected complex Randers space. Then, $(M, \tilde{F})$ is a generalized Berwald space if and only if $(\delta_k|\beta|)\eta^k = 0$.*

Theorem 3.3.2. *Let $(M, \tilde{F})$ be a connected complex Randers space. Then,*
i) α and $\tilde{F}$ are projectively related if and only if $\tilde{F}$ is generalized Berwald and $B^i = -P\eta^i$, for any $i = \overline{1, n}$, where $P = -\frac{\bar{\beta}}{2\tilde{F}|\beta|}\Gamma^k_{i\bar{j}}b_k\eta^i\bar{\eta}^j$. Moreover, the projective change is $\tilde{G}^i = \overset{a}{G^i}$.
ii) α is Kähler and α is projectively related to $\tilde{F}$ if and only if $\tilde{F}$ is a Kähler-Berwald metric.

Proof. We first prove i). Since α is pure Hermitian then is generalized Berwald. If α and $\tilde{F}$ are projectively related, then by Corollary 3.2.14 it results that $\tilde{F}$ is generalized Berwald. Thus,

by (3.28), (3.29) and Lemma 3.3.1, the conditions (3.23) are reduced to $B^i = -P\eta^i$, for any $i = \overline{1,n}$, where $P = -\frac{\bar{\beta}}{2\tilde{F}|\beta|}\Gamma^k_{i\bar{j}} b_k \eta^i \bar{\eta}^j$. Conversely, if $\tilde{F}$ is generalized Berwald, then the first condition from (3.23) is identically satisfied and by (3.29), it turns out that $B^i = -\frac{1}{\tilde{F}}\theta^{*l}(\dot{\partial}_l \tilde{F})\eta^i$ and $P = \frac{1}{\tilde{F}}\theta^{*i}(\dot{\partial}_i \tilde{F})$. All these conditions imply the projectivity of the metrics α and $\tilde{F}$.
ii) is a consequence of i), under assumptions that the metrics α and $\tilde{F}$ are Kähler, respectively. □

Example. Let $\Delta = \{(z,w) \in \mathbf{C}^2,\ |w| < |z| < 1\}$ be the Hartogs triangle with the pure Kähler-Hermitian metric

$$a_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\log \frac{1}{(1-|z|^2)(|z|^2-|w|^2)}),\ \alpha^2(z,w;\eta,\theta) = a_{i\bar{j}}\eta^i \bar{\eta}^j, \tag{3.30}$$

where $z,\ w, \eta,\ \theta$ are the local coordinates $z^1,\ z^2,\ \eta^1,\ \eta^2$, respectively, and $|z^i|^2 = z^i \bar{z}^i$, with $z^i \in \{z,w\}$, $\eta^i \in \{\eta,\theta\}$. We choose

$$b_z = \frac{w}{|z|^2-|w|^2},\ \ b_w = -\frac{z}{|z|^2-|w|^2}. \tag{3.31}$$

With these tools we have constructed in [28] the complex Randers metric $\tilde{F} = \alpha + |\beta|$, where $\alpha(z,w,\eta,\theta) = \sqrt{a_{i\bar{j}}(z,w)\eta^i\bar{\eta}^j}$ and $\beta(z,\eta) = b_i(z,w)\eta^i$. It is a Kähler-Berwald metric, and thus, by Theorem 3.3.2 ii), α and $\tilde{F}$ are projectively related.

Our second goal is to find when a complex Randers metric $\tilde{F} = \alpha + |\beta|$ on a domain D from $\mathbf{C}^n$ is projectively related to the complex Euclidean metric F on D.

To do this, we make several assumptions. First, we assume that $\tilde{F}$ is a Kähler-Berwald metric. Thus, by Theorem 3.3.2, ii) it turns out that α and $\tilde{F}$ are projectively related, α is Kähler and $\tilde{G}^i = \overset{a}{G^i}$. Next, we assume that α is projectively related to the Euclidean metric F. Therefore, Theorem 3.2.18 implies that $\overset{a}{G^i} = \frac{1}{\alpha}\frac{\partial \alpha}{\partial z^k}\eta^k\eta^i$. Under these statements, we compute

$$\begin{aligned}\frac{1}{\tilde{F}}\frac{\partial \tilde{F}}{\partial z^k}\eta^k\eta^i &= \frac{1}{\tilde{F}}\frac{\partial \alpha}{\partial z^k}\eta^k\eta^i + \frac{1}{\tilde{F}}\frac{\partial |\beta|}{\partial z^k}\eta^k\eta^i = \frac{1}{\tilde{F}}\frac{\partial \alpha}{\partial z^k}\eta^k\eta^i + \frac{1}{2|\beta|\tilde{F}}\left((\delta_k|\beta|)\eta^k + 2\bar{\beta}\overset{a}{G^l} b_l\right)\eta^i \\ &= \frac{\alpha}{\tilde{F}}\overset{a}{G^i} + \frac{|\beta|}{\tilde{F}}\frac{1}{\alpha}\frac{\partial \alpha}{\partial z^k}\eta^k\eta^i = \overset{a}{G^i}.\end{aligned}$$

Thus, $\tilde{G}^i = \frac{1}{\tilde{F}}\frac{\partial \tilde{F}}{\partial z^k}\eta^k\eta^i$, for any $i = \overline{1,n}$, which together with the Kähler-Berwald assumption for $\tilde{F}$, lead to the fact that $\tilde{F}$ is projectively related to the complex Euclidean metric F.

Conversely, by Theorem 3.2.18 it results that F and $\tilde{F}$ are projectively related if and only if the complex Randers metric $\tilde{F}$ is weakly Kähler and $\tilde{G}^i = \frac{1}{\tilde{F}}\frac{\partial \tilde{F}}{\partial z^k}\eta^k\eta^i$, for any $i = \overline{1,n}$. These induce that $\tilde{F}$ is generalized Berwald and morover, by Theorem 2.3.5, $\tilde{F}$ is a Kähler-Berwald metric. Now, taking into account Theorem 3.3.2, ii) it results that $\tilde{F}$ and α are projectively related, α is Kähler and $\tilde{G}^i = \overset{a}{G^i}$. So, we obtain

$$\overset{a}{G^i} = \frac{1}{\tilde{F}}\left(\frac{\partial \alpha}{\partial z^k}\eta^k + \frac{\bar{\beta}}{|\beta|}\overset{a}{G^l} b_l\right)\eta^i. \tag{3.32}$$

If we contract (3.32) with b_i it results the relation $\overset{a}{G^i} b_i = \frac{\beta}{\alpha}\frac{\partial\alpha}{\partial z^k}\eta^k$, which substituted into (3.32) yields $\overset{a}{G^i} = \frac{1}{\alpha}\frac{\partial\alpha}{\partial z^k}\eta^k\eta^i$, i.e. α is projectively related to the Euclidean metric F.

Therefore, the following theorem is proved

Theorem 3.3.3. *Let $\tilde{F} = \alpha + |\beta|$ be a complex Randers metric on a domain D from $\mathbf{C}^n$ and F the complex Euclidean metric on D. Then, F and $\tilde{F}$ are projectively related if and only if α is projectively related to the Euclidean metric F and $\tilde{F}$ is Kähler-Berwald.*

Chapter 4

Projective invariants of a complex Finsler space

This chapter, based on our papers [24, 23], extends the results presented in the previous chapter, exploring the projective change relationship of the complex Finsler metrics. It is a survey of the projective curvature invariants of Douglas and Weyl type which allow the complex Douglas space in relation to other special classes of complex Finsler spaces to be described.

4.1 Introduction and the main results

The subject of projective real Finsler spaces continues to be topical because of the projective curvature invariants that include Douglas curvature, Weyl curvature, generalized Douglas-Weyl curvature among others [53, 123, 128, 42, 109]. Exploration of these projective invariants leads to the special classes of metrics such as Douglas metrics and Finsler metrics of scalar flag curvature [46, 49, 69, 42, 128, 99, 43, 136]. A theorem by J. Douglas states that a Finsler space is projectively flat, if and only if, its Weyl and Douglas curvature invariants are zero. Some generalizations of the notion of Berwald space are strongly connected to the equation of geodesics, as pointed out by S. Bácsó and M. Matsumoto [46].

Recent years have seen evidence of significant progress in the study of complex Finsler geometry [7, 141, 144, 65, 160, 25, 26, 117]. Nevertheless, many subordinate subjects may be defined and studied. The prolongation of Chapter 3 aims to construct and investigate the projective invariants on complex Finsler manifolds, with a view to pointing out other applications of the complex projective change relationship. In [24] we stated the existence of the complex versions of the projective curvature invariants of Douglas and Weyl type and then, we clarified some notions related to these. There are some formal similarities with the real approach, but the differences between real and complex cases are more thorough. For example, there are three projective curvature invariants of Douglas type and the vanishing of these invariants characterizes the complex Douglas spaces. The study of complex Finsler metrics with constant holomorphic curvature is one most important problems in this geometry (see for instance [1, 90, 125, 144, 28]). Moreover, in [24], we proposed a characterization of the Kähler-Berwald spaces with constant holomorphic curvature, by means of a projective curvature invariant of Weyl type.

Subsequently, we make an overview of the chapter's content. In Section 4.2.1, we start by considering the structure equations satisfied by the connection form of the complex linear connection of Berwald type $B\Gamma$. Next, we have derived some of the Bianchi identities which specify the relations among the covariant derivatives of the curvature coefficients of $B\Gamma$. The first class of projective curvature invariants obtained by successive vertical differentiations of the projective change relationship is explored in Section 4.2.2. A direct consequence is the existence of three projective invariants of Douglas type and, by means of these, the complex Douglas spaces are defined. Additional investigations have to lead to some necessary and sufficient conditions that a complex Finsler space is of Douglas and generalized Kähler type, (Theorems 4.2.7 and 4.2.9, Corollary 4.2.12). Moreover, the complex Douglas property is preserved by the projective changes (Theorem 4.2.8). We have shown that any pure Hermitian complex Finsler space is a complex Douglas space. Thus, properly complex Douglas tensors are non-pure Hermitian quantities. The study of weakly Kähler projective changes is more significant (Section 4.2.3). We have proved that weakly Kähler Douglas spaces are Kähler-Berwald spaces (Theorem 4.2.17). A projective curvature invariant of Weyl type W^i_{jkh}, that formally looks the same as in the real case, is obtained. It vanishes in the Kähler context. For Kähler-Berwald spaces another projective curvature invariant of Weyl type $W^i_{j\bar{k}h}$ is found and moreover we have shown that $W^i_{j\bar{k}h} = 0$ if and only if the space is either pure Hermitian with the holomorphic curvature $\mathcal{K}_F$ equal to a constant value, or non-pure Hermitian with $\mathcal{K}_F = 0$ (Theorem 4.2.20).

Section 4.3 is devoted to the locally projectively flat complex Finsler metrics. The necessary and sufficient conditions for the locally projective flatness of a complex Finsler metric and other characterizations are established in Theorems 4.3.1, 4.3.4, 4.3.5 and Propositions 4.3.2, 4.3.3. At the end of this section, the locally projectively flat complex Finsler metrics are exemplified, illustrating better the interest for this work (Theorem 4.3.6).

In Section 4.4 a key detail for our arguments is that the equations of the complex geodesic curves can be rewritten in a more significant form, $\frac{d^2z^j}{dt^2}\eta^k - \frac{d^2z^k}{dt^2}\eta^j + 2D^{jk} = 0$. Consequently, the study of the complex Douglas spaces and their subclasses is reduced to the investigation of the functions D^{jk} derived from the equations of the geodesic curves. For example, some vertical differentiations of these function provide the tensors D^{jk}_{hrm} and $D^{jk}_{h\bar{l}m}$, which characterize the complex Douglas spaces (Theorems 4.4.2 and 4.4.3). Also, the holomorphicity of the functions D^{jk} is proper to the Kähler-Berwald spaces (Theorem 4.4.7).

The general theory on complex Douglas spaces is applied to the class of complex Randers spaces, in Section 4.5. Theorems 4.5.1 and 4.5.3 report on the necessary and sufficient conditions for a complex Randers metric $F = \alpha + |\beta|$ to be a complex Douglas metric. Namely, a complex Randers metric $F = \alpha + |\beta|$ is Douglas, if and only if, α and F are projectively related (Theorem 4.5.5). Moreover, a complex Randers-Douglas space of dimension two is a Kähler-Berwald space and the existence of the complex Randers-Douglas spaces is attested by some explicit examples for dimension $n \geq 3$.

4.2 Projective curvature invariants

Although the Chern-Finsler complex nonlinear connection (defined by the local coefficients (1.5)) is frequently used in complex Finsler geometry [1, 116], in this study, we use the canonical complex nonlinear connection because it derives from a complex spray, i.e. $\overset{c}{N^i_j} = \dot{\partial}_j G^i$ and $G^i = \frac{1}{2}N^i_j\eta^j$. Associated to the canonical complex nonlinear connection, we have the connection of Berwald type $B\Gamma$ which is our key tool for studying the projective changes of the complex Finsler metrics.

4.2.1 Curvature forms and Bianchi identities

Before of all, our goal is to describe the curvature forms and the Bianchi identities corresponding to $B\Gamma$. Let us to cosider the connection form $\omega_j^i(z,\eta) = G^i_{jk}dz^k + G^i_{j\bar{k}}d\bar{z}^k$ of $B\Gamma$, this satisfying the following structure equations

$$d(dz^i) - dz^k \wedge \omega_k^i = h\Omega^i, \quad d(\overset{c}{\delta}\eta^i) - \overset{c}{\delta}\eta^k \wedge \omega_k^i = v\Omega^i, \quad d\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i, \tag{4.1}$$

and their conjugates, where d is exterior differential with respect to the canonical ($c.n.c.$). Since

$$d(\overset{c}{\delta}\eta^i) = d\overset{c}{N^i_j} \wedge dz^j = \frac{1}{2}K^i_{jk}dz^k \wedge dz^j + \Theta^i_{j\bar{k}}d\bar{z}^k \wedge dz^j + G^i_{jk}\overset{c}{\delta}\eta^k \wedge dz^j + G^i_{j\bar{k}}\overset{c}{\delta}\bar{\eta}^k \wedge dz^j$$

and $G^i_{jk} = G^i_{kj}$, the torsion and curvature forms are given by the following formulas:

$$\begin{aligned}
h\Omega^i &= -G^i_{j\bar{k}}dz^j \wedge d\bar{z}^k,\\
v\Omega^i &= -\frac{1}{2}K^i_{jk}dz^j \wedge dz^k - \Theta^i_{j\bar{k}}dz^j \wedge d\bar{z}^k - G^i_{j\bar{k}}dz^j \wedge \overset{c}{\delta}\bar{\eta}^k - G^i_{j\bar{k}}\overset{c}{\delta}\eta^j \wedge d\bar{z}^k,\\
\Omega^i_j &= -\frac{1}{2}K^i_{jkh}dz^k \wedge dz^h - \frac{1}{2}K^i_{j\bar{k}\bar{h}}d\bar{z}^k \wedge d\bar{z}^h + K^i_{j\bar{h}k}dz^k \wedge d\bar{z}^h\\
&\quad - G^i_{jkh}dz^k \wedge \overset{c}{\delta}\eta^h - G^i_{j\bar{k}\bar{h}}d\bar{z}^k \wedge \overset{c}{\delta}\bar{\eta}^h - G^i_{j\bar{h}k}dz^k \wedge \overset{c}{\delta}\bar{\eta}^h + G^i_{j\bar{h}k}\overset{c}{\delta}\eta^k \wedge d\bar{z}^h,
\end{aligned}$$

where

$$K^i_{jk} = \overset{c}{\delta}_k\overset{c}{N^i_j} - \overset{c}{\delta}_j\overset{c}{N^i_k}, \;\; \Theta^i_{j\bar{k}} = \overset{c}{\delta}_{\bar{k}}\overset{c}{N^i_j} \;\; \text{and}$$

$$\begin{aligned}
K^i_{jkh} &= \overset{c}{\delta}_h G^i_{jk} - \overset{c}{\delta}_k G^i_{jh} + G^l_{jk}G^i_{lh} - G^l_{jh}G^i_{lk},\\
K^i_{j\bar{k}\bar{h}} &= \overset{c}{\delta}_{\bar{h}} G^i_{j\bar{k}} - \overset{c}{\delta}_{\bar{k}} G^i_{j\bar{h}} + G^l_{j\bar{k}}G^i_{l\bar{h}} - G^l_{j\bar{h}}G^i_{l\bar{k}},\\
K^i_{j\bar{k}h} &= \overset{c}{\delta}_h G^i_{j\bar{k}} - \overset{c}{\delta}_{\bar{k}} G^i_{jh} + G^l_{j\bar{k}}G^i_{lh} - G^l_{jh}G^i_{l\bar{k}}
\end{aligned}$$

are hh-, $\bar{h}\bar{h}$- and $h\bar{h}$- curvature tensors, respectively. We note that

$$G^i_{jkh} = \dot{\partial}_h G^i_{jk}, \;\; G^i_{j\bar{k}\bar{h}} = \dot{\partial}_{\bar{h}} G^i_{j\bar{k}}, \;\; G^i_{j\bar{k}h} = \dot{\partial}_h G^i_{j\bar{k}}$$

are hv-, $\bar{h}\bar{v}$- and $h\bar{v}$- curvature tensors, respectively. Moreover, they have the following properties

$$\begin{array}{l}
K^i_{jkh} = \dot{\partial}_j K^i_{kh}, \;\; K^i_{jkh}\eta^j = K^i_{kh}, \;\; K^i_{j\bar{k}\bar{h}} + K^i_{j\bar{h}\bar{k}} = 0,\\
G^i_{jkh}\eta^j = 0, \;\; G^i_{j\bar{k}h}\bar{\eta}^k = 0, \;\; G^i_{j\bar{k}h}\eta^j = G^i_{h\bar{k}}, \;\; G^i_{j\bar{k}\bar{h}}\bar{\eta}^h = -G^i_{j\bar{k}},\\
(\dot{\partial}_m G^i_{jkh})\eta^m = -G^i_{jkh}, \;\; (\dot{\partial}_m G^i_{j\bar{k}\bar{h}})\eta^m = G^i_{j\bar{k}\bar{h}}, \;\; (\dot{\partial}_m G^i_{j\bar{k}h})\eta^m = 0.
\end{array} \tag{4.2}$$

We mention that we preferred to denote by K^i_{jkh} the horizontal curvature tensors of $B\Gamma$, instead of classical real notation R^i_{jkh}. In this way, we avoid any confusion with the horizontal curvatures coefficients of the Chern-Finsler connection from (1.5).

Considering the exterior differential of the third structure equation from (4.1), it follows that

$$-\Omega_j^l \wedge \omega_l^i + \omega_j^l \wedge \Omega_l^i = d\Omega_j^i, \tag{4.3}$$

which leads to sixteen Bianchi identities. We mention here only some of these, which are needed for our proposed study

$$\begin{aligned} &\dot{\partial}_r G^i_{jkh} = \dot{\partial}_h G^i_{jkr},\ \dot{\partial}_r G^i_{j\bar{k}h} = \dot{\partial}_h G^i_{j\bar{k}r},\ \dot{\partial}_r G^i_{j\bar{k}\bar{h}} = \dot{\partial}_{\bar{h}} G^i_{j\bar{k}r}, \\ &\dot{\partial}_{\bar{r}} G^i_{jkh} = \dot{\partial}_h G^i_{j\bar{r}k},\ \dot{\partial}_{\bar{r}} G^i_{j\bar{k}\bar{h}} = \dot{\partial}_{\bar{h}} G^i_{j\bar{k}\bar{r}},\ \dot{\partial}_{\bar{r}} G^i_{j\bar{h}k} = \dot{\partial}_{\bar{h}} G^i_{j\bar{r}k}. \end{aligned} \tag{4.4}$$

When the space is generalized Berwald, the following identities are true:

$$\dot{\partial}_r K^i_{jkh} = 0, \quad \dot{\partial}_{\bar{r}} K^i_{jkh} = 0, \quad \dot{\partial}_r K^i_{j\bar{k}h} = 0, \quad \dot{\partial}_{\bar{r}} K^i_{j\bar{k}h} = 0$$

and for Kähler-Berwald spaces we get

$$K^i_{j\bar{r}k|\bar{h}} = K^i_{j\bar{h}k|\bar{r}}, \quad K^i_{j\bar{r}k|h} = K^i_{j\bar{r}h|k}, \tag{4.5}$$

where we denoted by '$_{|k}$ ' the horizontal covariant derivative with respect to Chern-Finsler connection.

4.2.2 Projective curvature invariants of Douglas type

As already mentioned in the previous chapter, the differential equations satisfied by a geodesic curve $z = z(s)$ of (M, F), with s a real parameter, are given by

$$\frac{d^2 z^i}{ds^2} + 2G^i(z(s), \frac{dz}{ds}) = \theta^{*i}(z(s), \frac{dz}{ds}), \tag{4.6}$$

where $z^i(s)$, $i = \overline{1,n}$, denote the coordinates along of the curve $z = z(s)$. We mention that $2G^i = N^i_j \eta^j = \overset{c}{N}{}^i_j \eta^j$ and $\theta^{*i} = 2g^{\bar{j}i} \overset{c}{\delta}_{\bar{j}} L$. The functions θ^{*i} are $(1,1)$-homogeneous, this is, $\theta^{*i}_k \eta^k = \theta^{*i}$ and $\theta^{*i}_{\bar{k}} \bar{\eta}^k = \theta^{*i}$, where $\theta^{*i}_k = \dot{\partial}_k \theta^{*i}$ and $\theta^{*i}_{\bar{k}} = \dot{\partial}_{\bar{k}} \theta^{*i}$. Moreover, the functions θ^{*i} hold the following relations:

$$\begin{aligned} &\theta^{*i}_{kj}\eta^k = 0,\ \theta^{*i}_{k\bar{h}}\eta^k = \theta^{*i}_{\bar{h}},\ \theta^{*i}_{\bar{k}j}\bar{\eta}^k = \theta^{*i}_j,\ \theta^{*i}_{\bar{k}\bar{h}}\bar{\eta}^k = 0, \\ &\theta^{*i}_{kjr}\eta^k = -\theta^{*i}_{jr},\ \theta^{*i}_{k\bar{h}j}\eta^k = 0,\ \theta^{*i}_{r\bar{k}j}\bar{\eta}^k = \theta^{*i}_{rj},\ \theta^{*i}_{j\bar{k}\bar{h}}\bar{\eta}^k = 0, \\ &\theta^{*i}_{k\bar{h}\bar{r}}\eta^k = \theta^{*i}_{\bar{h}\bar{r}},\ \theta^{*j}_{hlrm}\eta^m = -2\theta^{*j}_{hlr},\ \theta^{*j}_{h\bar{l}rm}\bar{\eta}^l = \theta^{*j}_{hrm}, \\ &\theta^{*j}_{h\bar{l}\bar{r}m}\bar{\eta}^l = 0,\ \theta^{*j}_{h\bar{l}rm}\eta^m = -\theta^{*j}_{h\bar{l}r},\ \theta^{*j}_{h\bar{l}\bar{r}m}\eta^m = 0, \end{aligned} \tag{4.7}$$

where the subscripts indicate differentiation with respect to η or $\bar{\eta}$, for example $\theta^{*i}_{kj} = \dot{\partial}_j \theta^{*i}_k = \dot{\partial}_k \theta^{*i}_j$, $\theta^{*i}_{r\bar{k}j} = \dot{\partial}_j \theta^{*i}_{r\bar{k}} = \dot{\partial}_r \theta^{*i}_{j\bar{k}} = \dot{\partial}_{\bar{k}} \theta^{*i}_{jr}$, $\theta^{*i}_{rm\bar{k}j} = \dot{\partial}_m \theta^{*i}_{j\bar{k}r}$, etc. We also emphasize the fact that $\theta^{*k}\eta_k = 0$, where $\eta_k = \dot{\partial}_k L$.

Let $\tilde{F}$ be another complex Finsler metric on the underlying complex manifold M. Corresponding to the metric $\tilde{F}$, we have the spray coefficients $\tilde{G}^i$ and the functions $\tilde{\theta}^{*i}$. According to Theorem 3.2.6, the complex Finsler metrics F and $\tilde{F}$ on M are projectively related if and only if there is a smooth function P on $T'M$, such that

$$\tilde{G}^i = G^i + B^i + P\eta^i,\ i = \overline{1,n}, \tag{4.8}$$

where $B^i = \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$, the relation (4.8) being called the projective change. An equivalent form for this is

$$\tilde{G}^i = G^i + V\eta^i \quad \text{and} \quad \tilde{\theta}^{*i} = \theta^{*i} + Q\eta^i, \quad i = \overline{1,n}, \tag{4.9}$$

where $V - \frac{1}{2}Q = P$, $V = (\dot{\partial}_k P)\eta^k$ is $(1,0)$-homogeneous and $Q = -2(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k$ is $(0,1)$-homogeneous.

Differentiating in (4.9) with respect to η^j, we get

$$\overset{c}{\tilde{N}}{}^i_j = \overset{c}{N}{}^i_j + V_j\eta^i + V\delta^i_j \quad \text{and} \quad \tilde{\theta}^{*i}_j = \theta^{*i}_j + Q_j\eta^i + Q\delta^i_j, \tag{4.10}$$

where $V_j = \dot{\partial}_j V$, $Q_j = \dot{\partial}_j Q$, $\tilde{\theta}^{*i}_j = \dot{\partial}_j \tilde{\theta}^{*i}$ and $\theta^{*i}_j = \dot{\partial}_j \theta^{*i}$. Thus, $V_j - \frac{1}{2}Q_j = P_j$, with $P_j = \dot{\partial}_j P$.

To eliminate V and Q from (4.10), we sum with $i = j$. Since $V_i\eta^i = V$ and $Q_i\eta^i = 0$, (4.10) gives

$$V = \frac{1}{n+1}(\overset{c}{\tilde{N}}{}^i_i - \overset{c}{N}{}^i_i) \ \text{ and } Q = \frac{1}{n}(\tilde{\theta}^{*i}_i - \theta^{*i}_i). \tag{4.11}$$

It follows that $P = \frac{1}{n+1}(\overset{c}{\tilde{N}}{}^i_i - \overset{c}{N}{}^i_i) - \frac{1}{2n}(\tilde{\theta}^{*i}_i - \theta^{*i}_i)$. Substituting this in (4.8), we find that the projective change is

$$\tilde{G}^i = G^i + \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i}) + \frac{1}{n+1}(\overset{c}{\tilde{N}}{}^l_l - \overset{c}{N}{}^l_l)\eta^i - \frac{1}{2n}(\tilde{\theta}^{*l}_l - \theta^{*l}_l)\eta^i, \ \ i = \overline{1,n}. \tag{4.12}$$

From here we can extract

$$D^i = G^i - \frac{1}{n+1}\overset{c}{N}{}^l_l\eta^i - \frac{1}{2}(\theta^{*i} - \frac{1}{n}\theta^{*l}_l\eta^i), \tag{4.13}$$

which are the components of a projective invariant, under the projective change (4.8).

Proposition 4.2.1. *Let (M, F) be a complex Finsler space. The functions D^i are the local coefficients of a complex spray if and only if F is weakly Kähler.*

Proof. First, D^i satisfy the rule (1.4), forasmuch as $\overset{c}{N}{}^l_l\eta^i$, θ^{*i} and $\theta^{*l}_l\eta^i$ have changes all like vectors. Second, D^i are $(2,0)$-homogeneous if and only if $\theta^{*i} = \frac{1}{n}\theta^{*l}_l\eta^i$. The last relation contracted by η_i gives $0 = \theta^{*i}\eta_i = \frac{1}{n}\theta^{*l}_l L$. Hence, $\theta^{*l}_l = 0$ and so $\theta^{*i} = 0$. □

Further on, the projective change (4.8) gives rise to various projective invariants. Indeed, some successive differentiations of (4.12) with respect to η and $\bar{\eta}$ yield

$$\begin{aligned}
\tilde{G}^i_{jkh} &= G^i_{jkh} + \frac{1}{n+1}[(\dot{\partial}_h \tilde{D}_{jk} - \dot{\partial}_h D_{jk})\eta^i + \sum_{(k,j,h)}(\tilde{D}_{jh} - D_{jh})\delta^i_k] \\
&\quad + \frac{1}{2}(\tilde{\theta}^{*i}_{jkh} - \theta^{*i}_{jkh}) - \frac{1}{2n}[(\dot{\partial}_h \tilde{\theta}^{*l}_{ljk} - \dot{\partial}_h \theta^{*l}_{ljk})\eta^i + \sum_{(j,k,h)}(\tilde{\theta}^{*l}_{ljh} - \theta^{*l}_{ljh})\delta^i_k], \\
\tilde{G}^i_{j\bar{k}\bar{h}} &= G^i_{j\bar{k}\bar{h}} + \frac{1}{n+1}[(\dot{\partial}_j \tilde{D}_{\bar{k}\bar{h}} - \dot{\partial}_j D_{\bar{k}\bar{h}})\eta^i + (\tilde{D}_{\bar{k}\bar{h}} - D_{\bar{k}\bar{h}})\delta^i_j] \\
&\quad + \frac{1}{2}(\tilde{\theta}^{*i}_{j\bar{k}\bar{h}} - \theta^{*i}_{j\bar{k}\bar{h}}) - \frac{1}{2n}[(\dot{\partial}_{\bar{h}} \tilde{\theta}^{*l}_{l\bar{k}j} - \dot{\partial}_{\bar{h}} \theta^{*l}_{l\bar{k}j})\eta^i + (\tilde{\theta}^{*l}_{l\bar{k}\bar{h}} - \theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j], \\
\tilde{G}^i_{j\bar{k}h} &= G^i_{j\bar{k}h} + \frac{1}{n+1}[(\dot{\partial}_h \tilde{D}_{\bar{k}j} - \dot{\partial}_h D_{\bar{k}j})\eta^i + (\tilde{D}_{\bar{k}j} - D_{\bar{k}j})\delta^i_h + (\tilde{D}_{\bar{k}h} - D_{\bar{k}h})\delta^i_j] \\
&\quad + \frac{1}{2}(\tilde{\theta}^{*i}_{j\bar{k}h} - \theta^{*i}_{j\bar{k}h}) - \frac{1}{2n}[(\dot{\partial}_h \tilde{\theta}^{*l}_{l\bar{k}j} - \dot{\partial}_h \theta^{*l}_{l\bar{k}j})\eta^i + (\tilde{\theta}^{*l}_{l\bar{k}j} - \theta^{*l}_{l\bar{k}j})\delta^i_h + (\tilde{\theta}^{*l}_{l\bar{k}h} - \theta^{*l}_{l\bar{k}h})\delta^i_j],
\end{aligned} \tag{4.14}$$

where $D_{kh} = G^i_{ikh}$, $D_{\bar{k}\bar{h}} = G^i_{i\bar{k}\bar{h}}$ and $D_{\bar{k}h} = G^i_{i\bar{k}h}$ are respectively, hv-, $\bar{h}\bar{v}$- and $h\bar{v}$- *Ricci tensors* and $\sum_{(j,k,h)}$ is the cyclic sum. From this we deduce the following three projective curvature invariants of *Douglas* type

$$D^i_{jkh} = G^i_{jkh} - \tfrac{1}{n+1}[(\dot{\partial}_h D_{jk})\eta^i + \textstyle\sum_{(k,j,h)} D_{jh}\delta^i_k] - \tfrac{1}{2}\{\theta^{*i}_{jkh} - \tfrac{1}{n}[(\dot{\partial}_h\theta^{*l}_{ljk})\eta^i + \textstyle\sum_{(j,k,h)}\theta^{*l}_{ljh}\delta^i_k]\},$$

$$D^i_{j\bar{k}\bar{h}} = G^i_{j\bar{k}\bar{h}} - \tfrac{1}{n+1}[(\dot{\partial}_j D_{\bar{k}\bar{h}})\eta^i + D_{\bar{k}\bar{h}}\delta^i_j] - \tfrac{1}{2}\{\theta^{*i}_{j\bar{k}\bar{h}} - \tfrac{1}{n}[(\dot{\partial}_{\bar{h}}\theta^{*l}_{l\bar{k}j})\eta^i + \theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j]\},$$

$$D^i_{j\bar{k}h} = G^i_{j\bar{k}h} - \tfrac{1}{n+1}[(\dot{\partial}_h D_{\bar{k}j})\eta^i + D_{\bar{k}j}\delta^i_h + D_{\bar{k}h}\delta^i_j] - \tfrac{1}{2}\{\theta^{*i}_{j\bar{k}h} - \tfrac{1}{n}[(\dot{\partial}_h\theta^{*l}_{l\bar{k}j})\eta^i + \theta^{*l}_{l\bar{k}j}\delta^i_h + \theta^{*l}_{l\bar{k}h}\delta^i_j]\}. \tag{4.15}$$

Definition 4.2.2. *A complex Finsler space (M, F) is called complex Douglas space if all of the invariants (4.15) are vanishing.*

Remark 4.2.3. *If F is Kähler-Berwald (i.e. $G^i_{jk}(z)$ and F is weakly Kähler, according to Theorem 3.2.2), then $G^i_{jkh} = G^i_{j\bar{k}\bar{h}} = G^i_{j\bar{k}h} = 0$ and $D_{kh} = D_{\bar{k}\bar{h}} = D_{\bar{k}h} = \theta^{*i} = 0$, and thus the projective curvature invariants of Douglas type are vanishing. It turns out that any Kähler-Berwald space is a complex Douglas space.*

Subsequently, the key of the proofs is the strong maximum principle which gives the independence of the fiber coordinates of the holomorphic and $(0,0)$-homogenous functions on $T'M$.

Lemma 4.2.4. *If one of hv-, $\bar{h}\bar{v}$- or $h\bar{v}$- Ricci tensors is vanishing then they are all vanishing.*

Proof. Supposing $D_{kh} = 0$, it results $G^i_{ikh} = 0$, which is equivalent with $\dot{\partial}_h G^i_{ik} = 0$. By conjugation, $\dot{\partial}_{\bar{h}} G^{\bar{i}}_{\bar{i}\bar{k}} = 0$, and so, $G^{\bar{i}}_{\bar{i}\bar{k}}$ are holomorphic with respect to η. But, $G^{\bar{i}}_{\bar{i}\bar{k}}$ are $(0,0)$-homogeneous and so they depend only on z, $(G^{\bar{i}}_{\bar{i}\bar{k}} = G^{\bar{i}}_{\bar{i}\bar{k}}(z))$. Hence, G^i_{ik} depend only on z and $\dot{\partial}_{\bar{h}} G^i_{ik} = D_{\bar{h}k} = 0$ which contracted by η^k give $\dot{\partial}_{\bar{h}} \overset{c}{N^i_i} = 0$, i.e. $G^i_{i\bar{h}} = 0$. So that, $D_{\bar{k}\bar{h}} = \dot{\partial}_{\bar{h}} G^i_{i\bar{k}} = 0$.

If $D_{\bar{k}\bar{h}} = 0$ then $\dot{\partial}_{\bar{h}} G^i_{i\bar{k}} = 0$ which contracted by $\bar{\eta}^h$ yield $G^i_{i\bar{k}} = 0$, because $G^i_{i\bar{k}\bar{h}}\bar{\eta}^h = -G^i_{i\bar{k}}$. It results $\dot{\partial}_{\bar{k}} G^i_{ih} = 0$, i.e. $D_{\bar{k}h} = 0$. Further on using the holomorphicity with respect to η and the homogeneity of the coefficients G^i_{ih} it results that G^i_{ih} depend on z alone. So, $\dot{\partial}_j G^i_{ih} = 0$ which give $D_{hj} = 0$.

If $D_{\bar{k}h} = 0$ then $\dot{\partial}_{\bar{k}} G^i_{ih} = 0$ and similarly it results that G^i_{ih} depend only on z and $G^i_{i\bar{k}} = 0$. This implies $D_{hj} = D_{\bar{k}\bar{h}} = 0$. □

Proposition 4.2.5. *Let (M, F) be a complex Finsler space. If $D^i_{j\bar{k}h} = 0$ then F is generalized Berwald and*

$$D^i_{jkh} = -\frac{1}{2}\{\theta^{*i}_{jkh} - \frac{1}{n}[(\dot{\partial}_h\theta^{*l}_{ljk})\eta^i + \sum_{(j,k,h)}\theta^{*l}_{ljh}\delta^i_k]\}, \tag{4.16}$$

$$D^i_{j\bar{k}\bar{h}} = -\frac{1}{2}\{\theta^{*i}_{j\bar{k}\bar{h}} - \frac{1}{n}[(\dot{\partial}_{\bar{h}}\theta^{*l}_{l\bar{k}j})\eta^i + \theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j]\},$$

$$\theta^{*i}_{j\bar{k}h} = \frac{1}{n}[(\dot{\partial}_h\theta^{*l}_{lj\bar{k}})\eta^i + \theta^{*l}_{l\bar{k}j}\delta^i_h + \theta^{*l}_{l\bar{k}h}\delta^i_j].$$

Proof. If $D^i_{j\bar{k}h}=0$ then

$$G^i_{j\bar{k}h}=\tfrac{1}{n+1}[(\dot{\partial}_h D_{\bar{k}j})\eta^i+D_{\bar{k}j}\delta^i_h+D_{\bar{k}h}\delta^i_j]+\tfrac{1}{2}\{\theta^{*i}_{j\bar{k}h}-\tfrac{1}{n}[(\dot{\partial}_h\theta^{*l}_{l\bar{k}j})\eta^i+\theta^{*l}_{l\bar{k}j}\delta^i_h+\theta^{*l}_{l\bar{k}h}\delta^i_j]\}. \tag{4.17}$$

which will be contracted by $\eta^j\eta^h$ and then by η_i.

Using $G^i_{j\bar{k}h}\eta^j\eta^h=G^i_{h\bar{k}}\,\eta^h=\dot{\partial}_{\bar{k}}G^i$, $(\dot{\partial}_j D_{\bar{k}h})\eta^j\eta^h=0$, $D_{\bar{k}h}\eta^h=G^l_{l\bar{k}}$ and taking into account (4.7), after the contraction by $\eta^j\eta^h$ of $G^i_{j\bar{k}h}$, we obtain

$$\dot{\partial}_{\bar{k}}G^i=\frac{2}{n+1}G^l_{l\bar{k}}\eta^i.$$

Due to Lemma 3.2.1, i.e. $(\dot{\partial}_{\bar{k}}G^i)\eta_i=0$, the contraction of the above relation with η_i leads to $G^l_{l\bar{k}}=0$. Its differential with respect to η^h gives $G^l_{l\bar{k}h}=0$, i.e. $D_{\bar{k}h}=0$ which plugged into (4.17) yields

$$G^i_{j\bar{k}h}=\frac{1}{2}\{\theta^{*i}_{j\bar{k}h}-\frac{1}{n}[(\dot{\partial}_h\theta^{*l}_{l\bar{k}j})\eta^i+\theta^{*l}_{l\bar{k}j}\delta^i_h+\theta^{*l}_{l\bar{k}h}\delta^i_j]\}.$$

The last relation contracted by η^j gives $G^i_{h\bar{k}}=0$. Next, it results $\dot{\partial}_{\bar{k}}G^i_{jh}=0$ which means that G^i_{jh} are holomorphic functions with respect to η. Together with their homogeneity it turns out $G^i_{jh}=G^i_{jh}(z)$. Hence $G^i_{jkh}=G^i_{j\bar{k}\bar{h}}=0$ and (4.16). □

Proposition 4.2.6. *Let (M,F) be a complex Finsler space. If $D^i_{j\bar{k}\bar{h}}=0$ then F is generalized Berwald and*

$$\begin{aligned}
D^i_{jkh}&=-\frac{1}{2}\{\theta^{*i}_{jkh}-\frac{1}{n}[(\dot{\partial}_h\theta^{*l}_{ljk})\eta^i+\sum_{(j,k,h)}\theta^{*l}_{ljh}\delta^i_k]\}, &(4.18)\\
\theta^{*i}_{j\bar{k}\bar{h}}&=\frac{1}{n}[(\dot{\partial}_{\bar{h}}\theta^{*l}_{lj\bar{k}})\eta^i+\theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j],\\
D^i_{j\bar{k}h}&=-\frac{1}{2}\{\theta^{*i}_{j\bar{k}h}-\frac{1}{n}[(\dot{\partial}_h\theta^{*l}_{lj\bar{k}})\eta^i+\theta^{*l}_{lj\bar{k}}\delta^i_h+\theta^{*l}_{l\bar{k}h}\delta^i_j]\}.
\end{aligned}$$

Proof. If $D^i_{j\bar{k}\bar{h}}=0$ then

$$G^i_{j\bar{k}\bar{h}}=\frac{1}{n+1}[(\dot{\partial}_j D_{\bar{k}\bar{h}})\eta^i+D_{\bar{k}\bar{h}}\delta^i_j]+\frac{1}{2}\{\theta^{*i}_{j\bar{k}\bar{h}}-\frac{1}{n}[(\dot{\partial}_{\bar{h}}\theta^{*l}_{lj\bar{k}})\eta^i+\theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j]\}. \tag{4.19}$$

The contraction of (4.19) by $\eta^j\bar{\eta}^h\eta_i$ and using (4.7) and $G^i_{j\bar{k}\bar{h}}\bar{\eta}^h\eta^j=-G^i_{j\bar{k}}\eta^j=-\dot{\partial}_{\bar{k}}G^i$, $D_{\bar{k}\bar{h}}\bar{\eta}^h=-G^i_{i\bar{k}}$, $(\dot{\partial}_j D_{\bar{k}\bar{h}})\bar{\eta}^h\eta^j=-(\dot{\partial}_j G^i_{i\bar{k}})\eta^j=-G^i_{i\bar{k}}$, lead to

$$0=-(\dot{\partial}_{\bar{k}}G^i)\eta_i=-\frac{2L}{n+1}G^l_{l\bar{k}},$$

which implies $G^l_{l\bar{k}}=0$ and so, $G^l_{l\bar{k}\bar{h}}=0$, i.e. $D_{\bar{k}\bar{h}}=0$. Plugging $D_{\bar{k}\bar{h}}=0$ into (4.19) we obtain

$$G^i_{j\bar{k}\bar{h}}=\frac{1}{2}\{\theta^{*i}_{j\bar{k}\bar{h}}-\frac{1}{n}[(\dot{\partial}_{\bar{h}}\theta^{*l}_{lj\bar{k}})\eta^i+\theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j]\}.$$

Now, the last relations contracted only by $\bar{\eta}^h$ leads to $G^i_{j\bar{k}}=0$. As above we obtain that G^i_{jh} depend only on z. So, the space is generalized Berwald and the relations (4.18) are true. □

Theorem 4.2.7. *Let (M,F) be a complex Finsler space. (M,F) is a complex Douglas space if and only if it is generalized Berwald with*

$$\theta^{*i}_{jkh} = \frac{1}{n}[(\dot{\partial}_h \theta^{*l}_{ljk})\eta^i + \sum_{(j,k,h)} \theta^{*l}_{ljh}\delta^i_k], \tag{4.20}$$
$$\theta^{*i}_{\bar{j}\bar{k}\bar{h}} = \frac{1}{n}[(\dot{\partial}_{\bar{h}} \theta^{*l}_{l\bar{j}\bar{k}})\eta^i + \theta^{*l}_{\bar{l}\bar{k}\bar{h}}\delta^i_{\bar{j}}],$$
$$\theta^{*i}_{\bar{j}kh} = \frac{1}{n}[(\dot{\partial}_h \theta^{*l}_{lj\bar{k}})\eta^i + \theta^{*l}_{lj\bar{k}}\delta^i_h + \theta^{*l}_{\bar{l}kh}\delta^i_{\bar{j}}]\}.$$

Proof. The direct implication is obvious by Propositions 4.2.5 and 4.2.6. Conversely, if the space is generalized Berwald, replacing the relations (4.20) into (4.15), it follows $D^i_{j\bar{k}h} = D^i_{jkh} = D^i_{j\bar{k}\bar{h}} = 0$. □

Let $\tilde{D}^i_{jkh}$, $\tilde{D}^i_{j\bar{k}\bar{h}}$ and $\tilde{D}^i_{j\bar{k}h}$ be the projective curvature invariants of Douglas type corresponding to the complex Finsler metric $\tilde{F}$. From (4.14) and (4.15) we immediately obtain the following result.

Theorem 4.2.8. *Let F and $\tilde{F}$ be projectively related complex Finsler metrics on M. F is a Douglas metric if and only if $\tilde{F}$ is also a Douglas metric.*

Since conditions (4.20) are checked for any weakly Kähler complex Finsler metric, the class of the complex Finsler spaces that fulfill (4.20) generalizes the class of the weakly Kähler complex Finsler spaces. We call this class as *generalized Kähler*.

The next theorem provides the necessary and sufficient conditions that a complex Finsler space to be generalized Kähler. We use the notation $K^i = \theta^{*i} - \frac{1}{n}\theta^{*l}_l\eta^i$.

Theorem 4.2.9. *Let (M,F) be a complex Finsler space. (M,F) is a generalized Kähler space if and only if the functions K^i are homogeneous polynomials in η and in $\bar{\eta}$ of first degree. Moreover, if the functions K^i vanish identically, then the space is weakly Kähler.*

Proof. We assume that the conditions (4.20) are checked. Due to (4.7), the contraction of these with η^j gives

$$\theta^{*i}_{kh} = \frac{1}{n}(\theta^{*l}_{lhk}\eta^i + \theta^{*l}_{lh}\delta^i_k + \theta^{*l}_{lk}\delta^i_h) \text{ and } \theta^{*i}_{\bar{k}\bar{h}} = \frac{1}{n}\theta^{*l}_{\bar{l}\bar{k}\bar{h}}\eta^i. \tag{4.21}$$

Differentiating (4.21) with respect to η and $\bar{\eta}$, it leads to (4.20). Thus, the sistems (4.20) and (4.21) are equivalent.

After two successive integrations with respect to $\bar{\eta}$ of the second formula in (4.21), using the homogeneity conditions, we have $\theta^{*i} = \frac{1}{n}\theta^{*l}_l\eta^i + f^i_{\bar{k}}\bar{\eta}^k$, where $f^i_{\bar{k}}(z,\bar{z},\eta)$. The subscript $f^i_{\bar{k}}$ does not mean differentiation with respect to $\bar{\eta}^k$ here. Since $f^i_{\bar{k}}$ depend only on z, $\bar{z}$ and η, then $f^i_{\bar{k}}\bar{\eta}^k$ are homogeneous polynomials in $\bar{\eta}$ of first degree.

It follows that $\theta^{*i}_{kh} = \frac{1}{n}(\theta^{*l}_{lhk}\eta^i + \theta^{*l}_{lh}\delta^i_k + \theta^{*l}_{lk}\delta^i_h) + \dot{\partial}_k\dot{\partial}_h(f^i_{\bar{r}}\bar{\eta}^r)$, which with the first formula in (4.21) implies $\dot{\partial}_k(\dot{\partial}_h(f^i_{\bar{r}}\bar{\eta}^r)) = 0$. Hence $\dot{\partial}_h(f^i_{\bar{r}}\bar{\eta}^r)$ does not depend on η and thus there exist the smooth functions $\varphi^i_{\bar{r}h}(z,\bar{z})$ such that $\dot{\partial}_h(f^i_{\bar{r}}\bar{\eta}^r) = \varphi^i_{\bar{r}h}\bar{\eta}^r$. Obviously, the subscripts in $\varphi^i_{\bar{r}h}$ do not indicate the differentiation with respect to $\bar{\eta}^r$ and η^h.

The last equation obtained can be rewritten as $\dot{\partial}_h(f^i_{\bar{r}}\bar{\eta}^r) = \dot{\partial}_h(\varphi^i_{\bar{r}s}\bar{\eta}^r\eta^s)$. Integration with respect to η gives $f^i_{\bar{r}}\bar{\eta}^r = \varphi^i_{\bar{r}s}\bar{\eta}^r\eta^s + \psi^i$, where $\psi^i(z,\bar{z},\bar{\eta})$ must be homogeneous polynomials in $\bar{\eta}$ of first degree.

This results in $\theta^{*i}=\frac{1}{n}\theta^{*l}_{l}\eta^i+\varphi^i_{\bar{r}s}\bar{\eta}^r\eta^s+\psi^i$, but the homogeneity properties of θ^{*i} and θ^{*l}_{l} lead to $\psi^i=0$. It still hold that $K^i=\varphi^i_{\bar{r}s}\bar{\eta}^r\eta^s$, that is K^i are the homogeneous polynomials in η and in $\bar{\eta}$ of first degree.

Conversely, if $K^i=\varphi^i_{\bar{r}s}\bar{\eta}^r\eta^s$, where $\varphi^i_{\bar{r}h}(z,\bar{z})$, then a few derivations of these conditions with respect to η and $\bar{\eta}$ yield (4.21).

In particular, if $K^i=0$, then $\theta^{*i}=\frac{1}{n}\theta^{*l}_{l}\eta^i$, which contracted with η_i gives $\theta^{*l}_{l}=0$ and thus $\theta^{*i}=0$. □

Some sufficient circumstances for the generalized Kähler property of a complex Finsler space are given in the following.

Proposition 4.2.10. *Let (M,F) be a complex Finsler space. If θ^{*i} are homogeneous polynomials in η and in $\bar{\eta}$ of first degree, then (M,F) is a generalized Kähler space.*

Proof. Under our assumption, we have $\theta^{*i}=f^i_{\bar{r}s}\bar{\eta}^r\eta^s$, where $f^i_{\bar{r}h}(z,\bar{z})$. Differentiating these with respect to η and $\bar{\eta}$ leads to either $\theta^{*i}_{kh}=\theta^{*i}_{\bar{k}\bar{h}}=0$ or $\theta^{*i}_{kh}=0$ and $\theta^{*i}_{\bar{k}h}$ depend only on z and $\bar{z}$. Thus, the conditions in (4.21) are fulfilled and the space is generalized Kähler. □

Examples of complex Douglas metrics are provided first by the class of pure Hermitian metrics. Considering the pure Hermitian metric $g_{i\bar{j}}=g_{i\bar{j}}(z)$, we obtain

$$G^i=\frac{1}{2}g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l\eta^j \text{ and } \theta^{*i}=-g^{\bar{m}i}(\frac{\partial g_{l\bar{m}}}{\partial \bar{z}^k}-\frac{\partial g_{l\bar{k}}}{\partial \bar{z}^m})\eta^l\bar{\eta}^k.$$

On one hand, since $\dot{\partial}_{\bar{h}}G^i=0$ any pure Hermitian metric is generalized Berwald. On the other hand, because the functions $g^{\bar{m}i}(\frac{\partial g_{l\bar{m}}}{\partial \bar{z}^k}-\frac{\partial g_{l\bar{k}}}{\partial \bar{z}^m})$ depend only on z and $\bar{z}$ we have that θ^{*i} are homogeneous polynomials in η and in $\bar{\eta}$ of first degree. According to Proposition 4.2.10, any pure Hermitian metric is then generalized Kähler.

Owing to Theorem 4.2.9 we have the following result.

Corollary 4.2.11. *Let (M,F) be a complex Finsler space. (M,F) is a weakly Kähler space if and only if the functions K^i vanish identically.*

In addition, by Theorem 4.2.9, an unsophisticated equivalent form of the Theorem 4.2.7 can be formulated.

Corollary 4.2.12. *Let (M,F) be a complex Finsler space. (M,F) is a complex Douglas space if and only if it is a generalized Berwald space and $K^i=\varphi^i_{\bar{r}s}\bar{\eta}^r\eta^s$, where $\varphi^i_{\bar{r}s}$ are smooth functions that depend only on z and $\bar{z}$.*

4.2.3 Weakly Kähler projective changes

The next discussion is focused on the weakly Kähler complex Finsler spaces. In this case, the projective invariants of Douglas type (4.15) are reduced to

$$\begin{aligned} D^i_{jkh}&=G^i_{jkh}-\frac{1}{n+1}[(\dot{\partial}_jD_{kh})\eta^i+\sum_{(j,k,h)}D_{jk}\delta^i_h], \\ D^i_{j\bar{k}\bar{h}}&=G^i_{j\bar{k}\bar{h}}-\frac{1}{n+1}[(\dot{\partial}_jD_{\bar{k}\bar{h}})\eta^i+D_{\bar{k}\bar{h}}\delta^i_j], \\ D^i_{j\bar{k}h}&=G^i_{j\bar{k}h}-\frac{1}{n+1}[(\dot{\partial}_jD_{\bar{k}h})\eta^i+D_{\bar{k}h}\delta^i_j+D_{\bar{k}j}\delta^i_h]. \end{aligned} \tag{4.22}$$

By Lemma 4.2.4, it immediately turns out the following result.

Proposition 4.2.13. *Let (M,F) be a weakly Kähler complex Finsler space. If one of hv-, $\bar{h}\bar{v}$- or $h\bar{v}$- Ricci tensors is vanishing, then*

$$D^i_{jkh}=G^i_{jkh},\ D^i_{j\bar{k}\bar{h}}=G^i_{j\bar{k}\bar{h}},\ D^i_{j\bar{k}h}=G^i_{j\bar{k}h}. \tag{4.23}$$

Proposition 4.2.14. *Let (M,F) be a weakly Kähler complex Finsler space. If one of statements (4.23) is true, then the hv-, $\bar{h}\bar{v}$- and $h\bar{v}$- Ricci tensors are vanishing.*

Proof. Suppose that $D^i_{j\bar{k}\bar{h}}=G^i_{j\bar{k}\bar{h}}$. Then, using (4.22) it results $(\dot{\partial}_j D_{\bar{k}\bar{h}})\eta^i+D_{\bar{k}\bar{h}}\delta^i_j=0$. Since $(\dot{\partial}_j D_{\bar{k}\bar{h}})\eta^j=D_{\bar{k}\bar{h}}$, hence $(n+1)D_{\bar{k}\bar{h}}=0$, and so $D_{\bar{k}\bar{h}}=0$. By Lemma 4.2.4, hv-, and $h\bar{v}$- Ricci tensors are vanishing. The proof is similar for $D^i_{jkh}=G^i_{jkh}$ or $D^i_{j\bar{k}h}=G^i_{j\bar{k}h}$. □

Corroborating (4.22) with Propositions 4.2.5 and 4.2.6, it follows

Corollary 4.2.15. *Let (M,F) be a weakly Kähler complex Finsler space.*
i) If $D^i_{j\bar{k}h}=0$ then $D^i_{jkh}=D^i_{j\bar{k}\bar{h}}=0$.
ii) If $D^i_{j\bar{k}\bar{h}}=0$ then $D^i_{jkh}=D^i_{j\bar{k}h}=0$.

Theorem 4.2.16. *Let (M,F) be a weakly Kähler complex Finsler space. If either $D^i_{j\bar{k}\bar{h}}=0$ or $D^i_{j\bar{k}h}=0$ then the space is Kähler-Berwald.*

Proof. If either $D^i_{j\bar{k}\bar{h}}=0$ or $D^i_{j\bar{k}h}=0$ then $G^i_{jh}=G^i_{jh}(z)$, which means that the space is generalized Berwald. The proof is completed by Theorem 3.2.2. □

Theorem 4.2.17. *If (M,F) is a complex weakly Kähler Douglas space then it is Kähler-Berwald.*

Proof. It results by Theorem 4.2.16. □

By Corollary 4.2.11 and Theorem 4.2.17, we have the next result.

Theorem 4.2.18. *If (M,F) is a complex Douglas space and the functions K^i vanish identically, then it is a Kähler-Berwald space.*

According to Theorem 3.2.8 (or [25, Theorem 3.2]), the weakly Kähler property is preserved by the projective changes, and moreover we have

$$\tilde{G}^i=G^i+P\eta^i, \tag{4.24}$$

where P is a $(1,0)$-homogeneous function. Under this simplified expression of the projective change, it turns out that

$$\begin{aligned}&\overset{c}{\tilde{N}}{}^i_j=\overset{c}{N}{}^i_j+P_j\eta^i+P\delta^i_j,\quad \overset{c}{\tilde{\delta}}_k=\overset{c}{\delta}_k-(P_k\eta^i+P\delta^i_k)\dot{\partial}_i,\\ &\tilde{G}^i_{jk}=G^i_{jk}+P_{jk}\eta^i+P_k\delta^i_j+P_j\delta^i_k,\quad \tilde{G}^i_{j\bar{k}}=G^i_{j\bar{k}}+P_{j\bar{k}}\eta^i+P_{\bar{k}}\delta^i_j,\end{aligned} \tag{4.25}$$

where $P_{jk}=\dot{\partial}_k P_j=P_{kj}$, $P_{\bar{k}}=\dot{\partial}_{\bar{k}}P$, $P_{j\bar{k}}=\dot{\partial}_{\bar{k}}P_j=\dot{\partial}_j P_{\bar{k}}$. In addition, the $(1,0)$ - homogeneity of P implies

$$P_k\eta^k=P,\ \ P_{j\bar{k}}\bar{\eta}^k=0,\ \ P_{jk}\eta^k=0,\ \ P_{\bar{k}}\bar{\eta}^k=0,\ \ P_{j\bar{k}}\eta^j=P_{\bar{k}}.$$

Hereinafter, we study the hh-curvatures tensor K^i_{jkh}. Under the projective change (4.24), we have

$$\begin{aligned}\tilde{K}^i_{kh} &= K^i_{kh} + \mathcal{A}_{(k,h)}[P_{k\overset{B}{|}h}\eta^i + (P_{\overset{B}{|}h} - PP_h)\delta^i_k],\\ \tilde{K}^i_{jkh} &= K^i_{jkh} + \mathcal{A}_{(k,h)}[P_{jk\overset{B}{|}h}\eta^i + P_{k\overset{B}{|}h}\delta^i_j + (P_{j\overset{B}{|}h} - P_jP_h - PP_{jh})\delta^i_k],\end{aligned} \tag{4.26}$$

where '$\overset{B}{|}_h$' is the horizontal covariant derivative with respect to $B\Gamma$ and $\mathcal{A}_{(k,h)}$ is the alternate operator, for example $\mathcal{A}_{(k,h)}\{P_{k\overset{B}{|}h}\} = P_{k\overset{B}{|}h} - P_{h\overset{B}{|}k}$. Next we make the following notations

$$X_{kh} = P_{k\overset{B}{|}h} - P_{h\overset{B}{|}k} \quad \text{and} \quad X_h = P_{\overset{B}{|}h} - PP_h$$

which have the properties

$$\begin{aligned}&\dot{\partial}_j X_h = P_{j\overset{B}{|}h} - P_jP_h - PP_{jh}, \;\; \dot{\partial}_j X_h - \dot{\partial}_h X_j = P_{j\overset{B}{|}h} - P_{h\overset{B}{|}j} = X_{jh},\\ &\dot{\partial}_j X_{kh} = P_{kj\overset{B}{|}h} - P_{hj\overset{B}{|}k}, \;\; (\dot{\partial}_j X_h)\eta^j = X_h, \;\; (\dot{\partial}_j X_{kh})\eta^j = 0,\\ &X_{kj}\eta^j = P_{k\overset{B}{|}0} - P_{\overset{B}{|}k} = X_{k0}.\end{aligned} \tag{4.27}$$

By means of these, the changes (4.26) become

$$\begin{aligned}\tilde{K}^i_{kh} &= K^i_{kh} + X_{kh}\eta^i + X_h\delta^i_k - X_k\delta^i_h,\\ \tilde{K}^i_{jkh} &= K^i_{jkh} + (\dot{\partial}_j X_{kh})\eta^i + X_{kh}\delta^i_j + (\dot{\partial}_j X_h)\delta^i_k - (\dot{\partial}_j X_k)\delta^i_h.\end{aligned} \tag{4.28}$$

Now, we introduce the hh- Ricci tensor $K_{kh} = K^i_{ikh}$. Another important tensor is $H_{jk} = K^i_{jki}$. The link between these horizontal curvature tensors is $H_{kj} - H_{jk} = K_{jk}$. Summing by $i = j$ and then $i = h$ together with a contraction by η^j, in the second relation from (4.28), it yields

$$\begin{aligned}X_{kh} &= \frac{1}{n+1}(\tilde{K}_{kh} - K_{kh}) = \frac{1}{n+1}[(\tilde{H}_{hk} - \tilde{H}_{kh}) - (H_{hk} - H_{kh})], \\ \tilde{H}_{0k} &= H_{0k} + X_{k0} - (n-1)X_k.\end{aligned} \tag{4.29}$$

From here, it turns out that

$$\begin{aligned}X_{k0} &= \frac{1}{n+1}[(\tilde{H}_{0k} - \tilde{H}_{k0}) - (H_{0k} - H_{k0})],\\ X_k &= -\frac{1}{n+1}(\tilde{H}_k - H_k), \;\; \text{with} \;\; H_k = \frac{1}{n-1}(nH_{0k} + H_{k0}),\end{aligned} \tag{4.30}$$

for any $n \geq 2$. Moreover,

$$K_{jk} = \dot{\partial}_j H_{k0} - \dot{\partial}_k H_{j0} = \dot{\partial}_k H_{0j} - \dot{\partial}_j H_{0k}, \text{ with } \; H_{jk} = \dot{\partial}_j H_{0k}. \tag{4.31}$$

Now, substituting (4.29) and (4.30) in (4.28) we obtain the following invariants

$$\begin{aligned}W^i_{kh} &= K^i_{kh} + \frac{1}{n+1}\mathcal{A}_{(k,h)}(H_{kh}\eta^i + H_h\delta^i_k),\\ W^i_{jkh} &= K^i_{jkh} + \frac{1}{n+1}\mathcal{A}_{(k,h)}[(\dot{\partial}_j H_{kh})\eta^i + H_{kh}\delta^i_j + (\dot{\partial}_j H_h)\delta^i_k],\end{aligned} \tag{4.32}$$

in which the second formula is a *projective curvature invariant of Weyl type.* We note that, if (M, F) is Kähler, then $W^i_{jkh} = 0$.

Theorem 4.2.19. *Let (M,F) be a weakly Kähler n-dimensional complex Finsler space, $n\geq 2$.*
i) $W^i_{jkh}=0$ if and only if $W^i_{kh}=0$;
ii) If $K_{kh}=0$ then $W^i_{jkh}=K^i_{jkh}+\frac{1}{n-1}(H_{jh}\delta^i_k-H_{jk}\delta^i_h)$;
iii) If $H_{kh}=0$ then $W^i_{jkh}=K^i_{jkh}$.

Proof. i) If $W^i_{jkh}=0$, then

$$K^i_{jkh}=-\frac{1}{n+1}\mathcal{A}_{(k,h)}[(\dot{\partial}_jH_{kh})\eta^i+H_{kh}\delta^i_j+(\dot{\partial}_jH_h)\delta^i_k],$$

which contracted by η^j give $K^i_{kh}=-\frac{1}{n+1}\mathcal{A}_{(k,h)}(H_{kh}\eta^i+H_h\delta^i_k)$ and hence, $W^i_{kh}=0$.

Conversely, if $W^i_{kh}=0$ then $K^i_{kh}=-\frac{1}{n+1}\mathcal{A}_{(k,h)}(H_{kh}\eta^i+H_h\delta^i_k)$. Differentiating with respect to η^j, it results

$$K^i_{jkh}=-\frac{1}{n+1}\mathcal{A}_{(k,h)}[(\dot{\partial}_jH_{kh})\eta^i+H_{kh}\delta^i_j+(\dot{\partial}_jH_h)\delta^i_k],$$

that is, $W^i_{jkh}=0$.
ii) If $K_{kh}=0$ then $H_{kj}=H_{jk}$. Substituting into (4.32) and using (4.30) and (4.31), it results our claim. iii) immediately results by (4.32) and (4.30). □

Aiming to obtain another projective curvature invariant of Weyl type, we assume that the weakly Kähler complex Finsler metric F is generalized Berwald. Therefore, we have $K^i_{j\bar{k}\bar{h}}=0$, $K^i_{j\bar{k}h}=-\delta_{\bar{k}}\overset{c}{G^i_{jh}}$ and by Bianchi identities, we get $\dot{\partial}_rK^i_{j\bar{k}h}=0$ and $\dot{\partial}_{\bar{r}}K^i_{j\bar{k}h}=0$.

We note that by a projective change, the generalized Berwald property of the metric L is transferred to the metric $\tilde{L}$. Moreover, the generalized Berwald property together with the weakly Kähler assumption implies that F and $\tilde{F}$ are Kähler-Berwald metrics (Theorem 3.2.2). Hence, $K^i_{j\bar{k}h}=-\delta_{\bar{k}}L^i_{jh}$. Therefore, under these assumptions, the function P from the projective change (4.24) is holomorphic with respect to η, i.e. $P_{\bar{k}}=0$ (see Proposition 3.2.16), and

$$\tilde{N}^i_j=N^i_j+P_j\eta^i+P\delta^i_j,\quad \tilde{\delta}_k=\delta_k-(P_k\eta^i+P\delta^i_k)\dot{\partial}_i, \tag{4.33}$$
$$\tilde{L}^i_{jk}=L^i_{jk}+P_{jk}\eta^i+P_k\delta^i_j+P_j\delta^i_k,\quad \tilde{G}^i_{j\bar{k}}=G^i_{j\bar{k}}=0.$$

Consequently,

$$\tilde{K}^i_{j\bar{k}h}=K^i_{j\bar{k}h}-P_{jh|\bar{k}}\eta^i-P_{j|\bar{k}}\delta^i_h-P_{h|\bar{k}}\delta^i_j, \tag{4.34}$$
$$0=P_{jhr|\bar{k}}\eta^i+P_{jh|\bar{k}}\delta^i_r+P_{jr|\bar{k}}\delta^i_h+P_{hr|\bar{k}}\delta^i_j.$$

Next, we consider the $h\bar{h}$-Ricci tensor $K_{\bar{k}h}=K^i_{i\bar{k}h}$. Since F is Kähler, $K^i_{i\bar{k}h}=K^i_{h\bar{k}i}$. Making $i=j$ in (4.34), it turns out

$$P_{h|\bar{k}}=-\frac{1}{n+1}(\tilde{K}_{\bar{k}h}-K_{\bar{k}h})\text{ and }P_{hr|\bar{k}}=0, \tag{4.35}$$

which substituted into the first equation from (4.34), give a new *projective curvature invariant of Weyl type,* which is valid only for the Kähler-Berwald spaces, namely

$$W^i_{j\bar{k}h}=K^i_{j\bar{k}h}-\frac{1}{n+1}(K_{\bar{k}j}\delta^i_h+K_{\bar{k}h}\delta^i_j). \tag{4.36}$$

We note that for any Kähler-Berwald space, the $h\bar{h}$-curvatures coefficients of Chern-Finsler connection can be rewritten as $R^i_{j\bar{k}h} = K^i_{j\bar{k}h} + K^l_{m\bar{k}h}\eta^m C^i_{jl}$. Thus,

$$R_{\bar{r}j\bar{k}h} = K_{\bar{r}j\bar{k}h} + K^l_{m\bar{k}h}\eta^m C_{j\bar{r}l},$$

where $K_{\bar{r}j\bar{k}h} = K^i_{j\bar{k}h} g_{i\bar{r}}$, and $R_{\bar{r}j\bar{k}h}\eta^j = K_{\bar{r}j\bar{k}h}\eta^j$. This implies that the holomorphic curvature of the Kähler-Berwald space (M, F) in direction η can be expressed as

$$\mathcal{K}_F(z, \eta) = \frac{2}{L^2} K_{\bar{r}j\bar{k}h}\bar{\eta}^r \eta^j \bar{\eta}^k \eta^h.$$

Theorem 4.2.20. *Let (M, F) be a connected n-dimensional Kähler-Berwald spaces, $n \geq 2$. Then, $W^i_{j\bar{k}h} = 0$ if and only if $K_{\bar{m}j\bar{k}h} = \frac{\mathcal{K}_F}{4}(g_{j\bar{k}}g_{h\bar{m}} + g_{h\bar{k}}g_{j\bar{m}})$. In this case, $\mathcal{K}_F = c$, where c is a constant on M and the space is either pure Hermitian with $K_{\bar{k}j} = \frac{c(n+1)}{4} g_{j\bar{k}}$ or non-pure Hermitian with $c = 0$ and $K^i_{j\bar{k}h} = 0$.*

Proof. Using (4.36) and $W^i_{j\bar{k}h} = 0$, it results

$$K^i_{j\bar{k}h} = \frac{1}{n+1}(K_{\bar{k}j}\delta^i_h + K_{\bar{k}h}\delta^i_j) \tag{4.37}$$

which contracted with $g_{i\bar{m}}$ gives

$$K_{\bar{m}j\bar{k}h} = \frac{1}{n+1}(K_{\bar{k}j}g_{h\bar{m}} + K_{\bar{k}h}g_{j\bar{m}}), \tag{4.38}$$

and

$$R_{\bar{m}j\bar{k}h} = \frac{1}{n+1}(K_{\bar{k}j}g_{h\bar{m}} + K_{\bar{k}h}g_{j\bar{m}} + K_{\bar{k}l}\eta^l C_{j\bar{m}h}), \tag{4.39}$$

where $C_{j\bar{m}h} = \dot{\partial}_h g_{j\bar{m}}$. Since $R_{\bar{r}j\bar{k}h} = R_{\bar{r}h\bar{k}j}$ (see [1, p. 105]), it results $R_{\bar{r}j\bar{k}h} = R_{\bar{k}j\bar{r}h}$, and therefore,

$$K_{\bar{r}j\bar{k}h}\eta^j = K_{\bar{k}j\bar{r}h}\eta^j. \tag{4.40}$$

From (4.38) also follows that $\mathcal{K}_F = \frac{4}{L(n+1)} K_{\bar{k}j}\eta^j\bar{\eta}^k$, which indeed, can be rewritten as $L\mathcal{K}_F = \frac{4}{n+1}K_{\bar{k}j}\eta^j\bar{\eta}^k$. Differentiating this last formula with respect to $\bar{\eta}^m$ and using again the Bianchi identity $\dot{\partial}_{\bar{m}} K_{\bar{k}h} = 0$, it follows that $\mathcal{K}_F \bar{\eta}_m + L(\dot{\partial}_{\bar{m}}\mathcal{K}_F) = \frac{4}{n+1}K_{\bar{m}j}\eta^j$. Now, due to (4.40), we obtain

$$\mathcal{K}_F \bar{\eta}_m = \frac{4}{n+1} K_{\bar{m}j}\eta^j. \tag{4.41}$$

Thus, $L(\dot{\partial}_{\bar{m}}\mathcal{K}_F) = 0$ and so, $\mathcal{K}_F$ depends only on z. Differentiating (4.41) with respect to η^l, it gives $K_{\bar{m}l} = \frac{(n+1)\mathcal{K}_F}{4} g_{l\bar{m}}$, which plugged into (4.38) yields $K_{\bar{m}j\bar{k}h} = \frac{\mathcal{K}_F}{4}(g_{j\bar{k}}g_{h\bar{m}} + g_{h\bar{k}}g_{j\bar{m}})$.

Conversely, since $K^i_{j\bar{k}h} = \frac{\mathcal{K}_F}{4}(g_{j\bar{k}}\delta^i_h + g_{h\bar{k}}\delta^i_j)$ and $K_{\bar{k}h} = \frac{(n+1)\mathcal{K}_F}{4} g_{h\bar{k}}$, the relation (4.36) implies $W^i_{j\bar{k}h} = 0$.

In order to prove that $\mathcal{K}_F$ is a constant on M we use the Bianchi identity $K^i_{j\bar{r}k|\bar{h}} = K^i_{j\bar{h}k|\bar{r}}$ from (4.5). Contracting by $g_{i\bar{m}}\bar{\eta}^m\eta^j\bar{\eta}^r\eta^k$, it gives

$$\mathcal{K}_{F|\bar{h}} = \frac{1}{L}\mathcal{K}_{F|\bar{0}}\bar{\eta}_h. \tag{4.42}$$

Taking into account $\mathcal{K}_{F|\bar{h}}|_j = \mathcal{K}_F|_{j|\bar{h}} = 0$, where '$|_k$ ' is the vertical covariant derivative with respect to Chern-Finsler connection, and differentiating (4.42), we easily deduce

$$0 = \mathcal{K}_{F|\bar{h}}|_j = \frac{1}{L}\mathcal{K}_{F|\bar{0}}(g_{j\bar{h}} - \frac{1}{L}\eta_j\bar{\eta}_h),$$

which multiplied by $g^{\bar{h}j}$, it turns out that $\frac{1}{L}(n-1)\mathcal{K}_{F|\bar{0}} = 0$. Plugging it into (4.42), it follows that $\mathcal{K}_{F|\bar{h}} = 0$, i.e. $\frac{\partial \mathcal{K}_F}{\partial \bar{z}^h} = 0$. By conjugation, $\frac{\partial \mathcal{K}_F}{\partial z^h} = 0$ and so, $\mathcal{K}_F$ is a constant c on M. This implies $K_{\bar{k}j} = \frac{c(n+1)}{4} g_{j\bar{k}}$ and its derivative with respect to η^l leads to $c(\dot{\partial}_l g_{j\bar{k}}) = 0$, and hence the last claim. □

4.3 Locally projectively flat complex Finsler metrics

The purpose of this section is to survey the locally projectively flat complex Finsler metrics.

Let $\tilde{L}$ be a locally Minkowski complex Finsler metric on the underlying manifold M. According to [5], this means that corresponding to $\tilde{L}$, at any point of M there exist local charts in which the fundamental metric tensor $\tilde{g}_{i\bar{j}}$ depends only on fiber coordinate η. Thus, it turns out that the spray coefficients $\tilde{G}^i = 0$ and the functions $\tilde{\theta}^{*i} = 0$ (in such local charts). The complex Finsler metrics L is called *locally projectively flat* if it is projectively related to the locally Minkowski metric $\tilde{L}$. Since the weakly Kähler property is preserved under the projective change, any locally projectively flat metric is weakly Kähler. Taking into account Theorem 3.2.10 (or [25, Theorem 3.3]), we state the following result.

Theorem 4.3.1. *L is locally projectively flat if and only if it is weakly Kähler and*

$$\dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l\tilde{L}) = 2P(\dot{\partial}_{\bar{r}}\tilde{L}), \quad r = \overline{1,n}, \tag{4.43}$$

where $P = \frac{1}{2\tilde{L}}(\delta_k\tilde{L})\eta^k$. Moreover, $G^i = -P\eta^i$.

Proof. The above equivalence results by Theorem 3.2.10 in which $\tilde{L}$ is a locally Minkowski metric on M. Taking into account $(\delta_k\tilde{L})\eta^k = -2G^l(\dot{\partial}_l\tilde{L})$, the condition (4.43) is equivalent to $-G^l\tilde{g}_{l\bar{r}} = P(\dot{\partial}_{\bar{r}}\tilde{L})$. By contraction with $\tilde{g}^{\bar{r}i}$, we obtain $G^i = -P\eta^i$. □

Proposition 4.3.2. *If L is locally projectively flat then $G^i = \frac{1}{2L}\frac{\partial L}{\partial z^k}\eta^k\eta^i$ and L is generalized Berwald.*

Proof. Since $G^i = \frac{1}{2}g^{\bar{m}i}\frac{\partial g_{r\bar{m}}}{\partial z^k}\eta^k\eta^r$ and L is locally projectively flat, then

$$g^{\bar{m}i}\frac{\partial g_{r\bar{m}}}{\partial z^k}\eta^k\eta^r = -2P\eta^i.$$

Contracting with η_i, it leads to $P = -\frac{1}{2L}\frac{\partial L}{\partial z^k}\eta^k$. So that, $G^i = \frac{1}{2L}\frac{\partial L}{\partial z^k}\eta^k\eta^i$.

Moreover, we have $g^{\bar{m}i}\frac{\partial g_{r\bar{m}}}{\partial z^k}\eta^k\eta^r = \frac{1}{L}\frac{\partial L}{\partial z^k}\eta^k\eta^i$, which implies

$$\dot{\partial}_{\bar{r}}G^i = -\frac{1}{2L^2}\frac{\partial L}{\partial z^k}\eta^k\eta^i\bar{\eta}_r + \frac{1}{2L}\frac{\partial^2 L}{\partial z^k\partial\bar{\eta}^r}\eta^k\eta^i = -\frac{1}{L}G^i\bar{\eta}_r + \frac{1}{2L}\frac{\partial g_{j\bar{r}}}{\partial z^k}\eta^j\eta^k\eta^i = -\frac{1}{L}G^i\bar{\eta}_r + \frac{1}{L}G^i\bar{\eta}_r = 0,$$

i.e. F is generalized Berwald. □

Proposition 4.3.3. *Let (M, F) be a complex Finsler space. If F is locally projectively flat then it is a Kähler-Berwald metric with $W^i_{j\bar{k}h} = 0$.*

Proof. By Proposition 4.3.2 and Theorem 3.2.2, it follows that L is a complex Berwald metric. Since $\tilde{K}^i_{j\bar{k}h} = \tilde{K}_{\bar{k}h} = 0$, the relations (4.34) and (4.35), give $K^i_{j\bar{k}h} = \frac{1}{n+1}(K_{\bar{k}j}\delta^i_h + K_{\bar{k}h}\delta^i_j)$ and so, $W^i_{j\bar{k}h} = 0$. □

By Theorem 4.2.20 we have proved the following result.

Theorem 4.3.4. *Let (M,F) be a connected n-dimensional complex Finsler space, $n \geq 2$. If F is locally projectively flat then it is of constant holomorphic curvature. Moreover, if the constant value of the holomorphic curvature is non-zero, then (M,F) is a pure Hermitian space.*

Some refinements of the above results on locally projectively flat metrics we obtained in [14, Theorem 2.4]. Namely, if F is a Kähler-Berwald metric on domain D from $\mathbb{C}^n$ with $G^i = \frac{1}{F}\frac{\partial F}{\partial z^k}\eta^k\eta^i$, then according to Theorem 3.2.18, it is projectively related to the standard Euclidean metric on D, and thus it is locally projectively flat. Conversely, if F is locally projectively flat, then by Propositions 4.3.2 and 4.3.2 it is Kähler-Berwald and $G^i = \frac{1}{F}\frac{\partial F}{\partial z^k}\eta^k\eta^i$. So, we have justified the following result.

Theorem 4.3.5. *Let F be a complex Finsler metric on domain D from $\mathbb{C}^n$. F is locally projectively flat if and only if it is Kähler-Berwald and $G^i = \frac{1}{F}\frac{\partial F}{\partial z^k}\eta^k\eta^i$.*

Next we study as an application the weakly Kähler complex Finsler metrics L with the spray coefficients $G^i = \rho_r\eta^r\eta^i$, where ρ is a smooth complex function depending only on $z \in M$, $\rho_r = \frac{\partial \rho}{\partial z^r}$ and $\rho_{r\bar{h}} = \frac{\partial \rho_r}{\partial \bar{z}^h}$ is Hermitian, i.e. $\overline{\rho_{r\bar{h}}} = \rho_{h\bar{r}}$, and it is nondegenerated.

Theorem 4.3.6. *Let (M,F) be a connected weakly Kähler n-dimensional complex Finsler space, $n \geq 2$, with $G^i = \rho_r\eta^r\eta^i$. Then,*
i) L is locally projectively flat;
ii) L is a Kähler-Berwald metric;
iii) L is a pure Hermitian of non-zero constant holomorphic curvature $\mathcal{K}_F = -\frac{4}{L}\rho_{r\bar{h}}\eta^r\bar{\eta}^h$.
iv) ρ satisfies the differential equations

$$\rho_{r\bar{h}k} = \rho_r\rho_{k\bar{h}} + \rho_k\rho_{r\bar{h}}, \tag{4.44}$$

where $\rho_{r\bar{h}k} = \frac{\partial \rho_{r\bar{h}}}{\partial z^k} = \frac{\partial \rho_{k\bar{h}}}{\partial z^r} = \frac{\partial \rho_{rk}}{\partial \bar{z}^h}$ and $\rho_{r\bar{h}k} = \rho_{k\bar{h}r}$

Proof. In order to prove i), we use Theorem 4.3.1. Let $\tilde{L}$ be a locally Minkowski metric on M. Since L is weakly Kähler, we must show only that the equation (4.43) is satisfied. Indeed, we have $\dot{\partial}_{\bar{r}}G^l = 0$, $(\delta_k\tilde{L})\eta^k = -2G^l(\dot{\partial}_l\tilde{L}) = -2\rho_r\eta^r\eta^l(\dot{\partial}_l\tilde{L}) = -2\tilde{L}\rho_r\eta^r$, and thus $\dot{\partial}_{\bar{r}}(\delta_k\tilde{L})\eta^k = -2(\dot{\partial}_{\bar{r}}\tilde{L})\rho_l\eta^l$, which implies the equation (4.43).
Since $\dot{\partial}_{\bar{r}}G^l = 0$, L is generalized Berwald. Thus, Theorem 3.2.2 yields ii).
iii) Theorem 4.3.4 together with i) and ii) show that $W^i_{j\bar{k}h} = 0$ and L is of constant holomorphic curvature. Since L is a Kähler-Berwald metric, $\delta_{\bar{k}} = \overset{c}{\delta}_{\bar{k}}$ and $L^i_{jh} = G^i_{jh}$. Hence $K^i_{j\bar{k}h} = -\delta_{\bar{k}}L^i_{jh}$, which will be rewritten in terms of derivatives of ρ. Indeed, two successive differentiations of the equations $G^i = \rho_r\eta^r\eta^i$ lead to $L^i_{jk} = \rho_k\delta^i_j + \rho_j\delta^i_k$. Consequently, $K^i_{j\bar{k}h} = -\rho_{j\bar{k}}\delta^i_h - \rho_{h\bar{k}}\delta^i_j$ which gives $K_{\bar{r}j\bar{k}h} = -\rho_{j\bar{k}}g_{h\bar{r}} - \rho_{h\bar{k}}g_{j\bar{r}}$ and thus,

$$\mathcal{K}_F = -\frac{4}{L}\rho_{r\bar{h}}\eta^r\bar{\eta}^h. \tag{4.45}$$

Since $\rho_{r\bar{h}}$ is nondegenerated, $\mathcal{K}_F \neq 0$ and by Theorem 4.3.4 it results that L is a pure Hermitian metric.
iv) To justify that ρ satisfies (4.44), we use (4.45). This implies

$$L = -\frac{4}{\mathcal{K}_F}\rho_{r\bar{h}}\eta^r\bar{\eta}^h = g_{r\bar{h}}\eta^r\bar{\eta}^h, \tag{4.46}$$

which gives

$$g_{r\bar{h}} = -\frac{4}{\mathcal{K}_F}\rho_{r\bar{h}} \text{ and } \delta_k g_{r\bar{h}} = -\frac{4}{\mathcal{K}_F}\rho_{r\bar{h}k}. \tag{4.47}$$

Now, using (1.5) and $L^i_{jk} = \rho_k\delta^i_j + \rho_j\delta^i_k$, it turns out that

$$\delta_k g_{j\bar{m}} = \rho_k g_{j\bar{m}} + \rho_j g_{k\bar{m}} \tag{4.48}$$

The substitution of (4.47) into (4.48) implies (4.44). Moreover, the Kähler property of L gives $\rho_{r\bar{h}k} = \rho_{k\bar{h}r}$. □

4.4 Some results on complex Douglas spaces

Considering $z = z(s)$ a geodesic curve of (M, F), it satisfies (5.29). Taking an arbitrary transformation of the parameter $t = t(s)$, with $\frac{dt}{ds} > 0$, the equations (5.29) are generally not preserved. Indeed, for the new parameter t we have the following relations $\frac{dz^i}{ds} = \frac{dz^i}{dt}\frac{dt}{ds}$, $\frac{d^2z^i}{ds^2} = \frac{d^2z^i}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{dz^i}{dt}\frac{d^2t}{ds^2}$ and $\theta^{*k}\left(z, \frac{dz}{ds}\right) = \left(\frac{dt}{ds}\right)^2\theta^{*k}(z, \frac{dz}{dt})$, which yield that

$$\left[\frac{d^2z^i}{dt^2} + 2G^i(z, \tfrac{dz}{dt}) - \theta^{*i}(z, \tfrac{dz}{dt})\right]\left(\frac{dt}{ds}\right)^2 = \frac{d^2z^i}{ds^2} - \frac{dz^i}{dt}\frac{d^2t}{ds^2} + 2G^i(z, \tfrac{dz}{ds}) - \theta^{*i}(z, \tfrac{dz}{ds}) = -\frac{dz^i}{dt}\frac{d^2t}{ds^2}.$$

Therefore, the equations (5.29) in t parameter became,

$$\frac{d^2z^i}{dt^2} + 2G^i(z(t), \frac{dz}{dt}) = \theta^{*i}(z(t), \frac{dz}{dt}) - \frac{dz^i}{dt}\frac{d^2t}{ds^2}\frac{1}{\left(\frac{dt}{ds}\right)^2},\ i = \overline{1, n}, \tag{4.49}$$

which are equivalent to

$$\frac{\frac{d^2z^i}{dt^2} + 2G^i(z, \frac{dz}{dt}) - \theta^{*i}(z, \frac{dz}{dt})}{\frac{dz^i}{dt}} = -\frac{d^2t}{ds^2}\frac{1}{\left(\frac{dt}{ds}\right)^2},\ i = \overline{1, n}. \tag{4.50}$$

We can rewrite (4.50), taking two different values for i, as

$$\frac{\frac{d^2z^j}{dt^2} + 2G^j(z, \frac{dz}{dt}) - \theta^{*j}(z, \frac{dz}{dt})}{\frac{dz^j}{dt}} = \frac{\frac{d^2z^k}{dt^2} + 2G^k(z, \frac{dz}{dt}) - \theta^{*k}(z, \frac{dz}{dt})}{\frac{dz^k}{dt}} = -\frac{d^2t}{ds^2}\frac{1}{\left(\frac{dt}{ds}\right)^2}, \tag{4.51}$$

for any $j, k = \overline{1, n}$, and the first equation in (4.51) leads to

$$\frac{d^2z^j}{dt^2}\frac{dz^k}{dt} - \frac{d^2z^k}{dt^2}\frac{dz^j}{dt} + 2G^j\frac{dz^k}{dt} - 2G^k\frac{dz^j}{dt} = \theta^{*i}\frac{dz^k}{dt} - \theta^{*k}\frac{dz^i}{dt}. \tag{4.52}$$

Since $\eta^k = \frac{dz^k}{dt}$, along the geodesic curve $z = z(t(s))$ of (M, F) the differential equations hold

$$\frac{d^2z^j}{dt^2}\eta^k - \frac{d^2z^k}{dt^2}\eta^j + 2D^{jk} = 0, \tag{4.53}$$

where $D^{jk} = G^j\eta^k - G^k\eta^j - \frac{1}{2}(\theta^{*j}\eta^k - \theta^{*k}\eta^j)$.

The homogeneity property of the spray coefficients G^i and of the functions θ^{*i} leads to

$$D^{jk}_r\eta^r + D^{jk}_{\bar{r}}\bar{\eta}^r = 3D^{jk} \quad \text{and} \quad D^{jk}_{\bar{r}}\bar{\eta}^r = -\frac{1}{2}(\theta^{*j}\eta^k - \theta^{*k}\eta^j), \tag{4.54}$$

where $D^{jk}_r = \dot{\partial}_r D^{jk}$ and $D^{jk}_{\bar{r}} = \dot{\partial}_{\bar{r}} D^{jk}$. Furthermore, differentiating (4.54) with respect to η gives

$$\begin{aligned} &D^{jk}_{rh}\eta^r + D^{jk}_{\bar{r}h}\bar{\eta}^r = 2D^{jk}_h, \quad D^{jk}_{rhl}\eta^r + D^{jk}_{\bar{r}hl}\bar{\eta}^r = D^{jk}_{hl},\\ &D^{jk}_{rhlm}\eta^r + D^{jk}_{\bar{r}hlm}\bar{\eta}^r = 0, \quad D^{jk}_{\bar{m}rhl}\eta^r + D^{jk}_{\bar{m}\bar{r}hl}\bar{\eta}^r = 0, \end{aligned} \tag{4.55}$$

where $D^{jk}_{rh} = \dot{\partial}_h D^{jk}_r$, $D^{jk}_{\bar{r}h} = \dot{\partial}_h D^{jk}_{\bar{r}}$, $D^{jk}_{rhl} = \dot{\partial}_l D^{jk}_{rh}$, and all that. We note that the sequence of the subscripts does not matter; for example D^{jk}_{rhl} is same with D^{jk}_{rlh}.

Successive differentiations of D^{jk} with respect to η or $\bar{\eta}$ yield the tensors

$$\begin{aligned} D^{jk}_{hlrm} &= (\dot{\partial}_m G^j_{hlr})\eta^k + G^j_{hlr}\delta^k_m + G^j_{hlm}\delta^k_r + G^j_{hmr}\delta^k_l + G^j_{mlr}\delta^k_h + \theta^{*j}_{hlrm}\eta^k\\ &\quad + \theta^{*j}_{hlr}\delta^k_m + \theta^{*j}_{hlm}\delta^k_r + \theta^{*j}_{hmr}\delta^k_l + \theta^{*j}_{mlr}\delta^k_h - [j,k],\\ D^{jk}_{hl\bar{r}m} &= (\dot{\partial}_m G^j_{hl\bar{r}})\eta^k + G^j_{hl\bar{r}}\delta^k_m + G^j_{ml\bar{r}}\delta^k_h + \theta^{*j}_{hl\bar{r}m}\eta^k + \theta^{*j}_{hl\bar{r}}\delta^k_m + \theta^{*j}_{ml\bar{r}}\delta^k_h - [j,k],\\ D^{jk}_{h\bar{l}rm} &= (\dot{\partial}_m G^j_{h\bar{l}r})\eta^k + G^j_{h\bar{l}r}\delta^k_m + G^j_{h\bar{l}m}\delta^k_r + G^j_{m\bar{l}r}\delta^k_h + \theta^{*j}_{h\bar{l}rm}\eta^k\\ &\quad + \theta^{*j}_{h\bar{l}r}\delta^k_m + \theta^{*j}_{h\bar{l}m}\delta^k_r + \theta^{*j}_{m\bar{l}r}\delta^k_h - [j,k], \end{aligned} \tag{4.56}$$

where $[j,k]$ denotes interchanges of indices j and k of the preceding terms.

Lemma 4.4.1. *The tensors D^{jk}_{hrm} and $D^{jk}_{h\bar{l}m}$ depend only on z and $\bar{z}$ if and only if $D^{jk}_{h\bar{l}rm} = D^{jk}_{hl\bar{r}m} = 0$. Moreover, given any of them, $D^{jk}_{rhlm} = 0$.*

Proof. The direct implication is obvious. Conversely, $D^{jk}_{h\bar{l}rm} = D^{jk}_{hl\bar{r}m} = 0$ with (4.55) implies that D^{jk}_{hrm} and $D^{jk}_{h\bar{l}m}$ are holomorphic with respect to η and $(0,0)$-homogeneous. Thus, applying the strong maximum principle (see [26]), it results $D^{jk}_{hrm}(z,\bar{z})$ and $D^{jk}_{h\bar{l}m}(z,\bar{z})$. Any of these gives $D^{jk}_{rhlm} = 0$. □

Theorem 4.4.2. *Let (M,F) be a complex Finsler space. (M,F) is Douglas if and only if it is generalized Berwald with D^{jk}_{hrm} and $D^{jk}_{h\bar{l}m}$ depending only on z and $\bar{z}$.*

Proof. Whether or not the space is Douglas, according to Theorem 4.2.7, it is generalized Berwald with (4.20). Using the fact that $G^i_{jk}(z,\bar{z})$ (i.e. the generalized Berwald property), it follows that

$$\begin{aligned} D^{jk}_{hlrm} &= \theta^{*j}_{hlrm}\eta^k + \theta^{*j}_{hlr}\delta^k_m + \theta^{*j}_{hlm}\delta^k_r + \theta^{*j}_{hmr}\delta^k_l + \theta^{*j}_{mlr}\delta^k_h - [j,k],\\ D^{jk}_{hl\bar{r}m} &= \theta^{*j}_{hl\bar{r}m}\eta^k + \theta^{*j}_{hl\bar{r}}\delta^k_m + \theta^{*j}_{ml\bar{r}}\delta^k_h - [j,k],\\ D^{jk}_{h\bar{l}rm} &= \theta^{*j}_{h\bar{l}rm}\eta^k + \theta^{*j}_{h\bar{l}r}\delta^k_m + \theta^{*j}_{h\bar{l}m}\delta^k_r + \theta^{*j}_{m\bar{l}r}\delta^k_h - [j,k]; \end{aligned} \tag{4.57}$$

Substituting (4.20) into (4.57), then yields $D^{jk}_{hlrm} = D^{jk}_{hl\bar{r}m} = D^{jk}_{h\bar{l}rm} = 0$ and thus, $D^{jk}_{hrm}(z,\bar{z})$ and $D^{jk}_{h\bar{l}m}(z,\bar{z})$.

Conversely, under assumption of generalized Berwald the tensors D^{jk}_{hlrm}, $D^{jk}_{h\bar{l}\bar{r}m}$ and $D^{jk}_{h\bar{l}rm}$ are given by (4.57). Summing in these by $k = m$ and bearing (4.7), we obtain

$$\begin{aligned} D^{jk}_{hlrk} &= n\{\theta^{*j}_{hlr} - \frac{1}{n}[(\dot{\partial}_h\theta^{*k}_{klr})\eta^j + \sum_{(k,h,l)} \theta^{*k}_{khl}\delta^j_r]\}, \\ D^{jk}_{h\bar{l}\bar{r}k} &= n\{\theta^{*j}_{h\bar{l}\bar{r}} - \frac{1}{n}[(\dot{\partial}_{\bar{r}}\theta^{*k}_{kh\bar{l}})\eta^j + \theta^{*k}_{k\bar{l}\bar{r}}\delta^j_h]\}, \\ D^{jk}_{h\bar{l}rk} &= n\{\theta^{*j}_{h\bar{l}r} - \frac{1}{n}[(\dot{\partial}_h\theta^{*k}_{kr\bar{l}})\eta^j + \theta^{*k}_{kh\bar{l}}\delta^j_r + \theta^{*k}_{kr\bar{l}}\delta^j_h]\}, \end{aligned}$$

which with (4.15), lead to

$$D^{jk}_{hlrk} = (n+1)D^j_{hlr}, \quad D^{jk}_{h\bar{l}\bar{r}k} = (n+1)D^j_{h\bar{l}\bar{r}}, \quad D^{jk}_{h\bar{l}rk} = (n+1)D^j_{h\bar{l}r}. \tag{4.58}$$

However, under our assumptions we also have $D^{jk}_{rhlm} = D^{jk}_{h\bar{l}rm} = D^{jk}_{h\bar{l}\bar{r}m} = 0$. These, along with (4.58) prove that the space is Douglas. □

Theorem 4.4.3. *Let (M, F) be a complex Finsler space. (M, F) is Douglas if and only if it is generalized Kähler with D^{jk}_{hrm} and $D^{jk}_{h\bar{l}m}$ depending only on z and $\bar{z}$.*

Proof. The direct implication follows from Theorems 4.2.7 and 4.4.2. Conversely, under assumption that the space is generalized Kähler, the conditions (4.20) are identically checked and $D^{jk}_j = -(n+1)G^k + \overset{c}{N}{}^j_j\eta^k - \varphi^k_{\bar{r}s}\bar{\eta}^r\eta^s$, where $\varphi^k_{\bar{r}s}$ are smooth functions which depends only on z and $\bar{z}$. After three derivations of these last relations with respect to η or $\bar{\eta}$, it follows that

$$\begin{aligned} D^{jk}_{jrhm} &= -(n+1)G^i_{rhm} + (\dot{\partial}_m D_{rh})\eta^k + \sum_{(j,h,k)} D_{jh}\delta^i_k, \\ D^{jk}_{jr\bar{h}\bar{m}} &= -(n+1)G^i_{r\bar{h}\bar{m}} + (\dot{\partial}_r D_{\bar{h}\bar{m}})\eta^k + D_{\bar{h}\bar{m}}\delta^k_r, \\ D^{jk}_{jr\bar{h}m} &= -(n+1)G^i_{r\bar{h}m} + (\dot{\partial}_r D_{\bar{h}m})\eta^k + D_{\bar{h}r}\delta^k_m + D_{\bar{h}m}\delta^k_r. \end{aligned}$$

Our assumptions $D^{jk}_{hrm}(z,\bar{z})$ and $D^{jk}_{h\bar{l}m}(z,\bar{z})$, along with the conditions (4.20) and (4.15), then lead to $D^i_{jkh} = D^i_{j\bar{k}\bar{h}} = D^i_{j\bar{k}h} = 0$, that is, the space is Douglas. □

Lemma 4.4.4. *Let (M, F) be a complex Finsler space. Then, $G^i = P\eta^i$, where P is a smooth function on $T'M$, if and only if $D^{jk} = -\frac{1}{2}(\theta^{*j}\eta^k - \theta^{*k}\eta^j)$. Moreover, given any of them, the functions D^{jk} are $(2,1)$-homogeneous and (M, F) is a generalized Berwald space.*

Proof. Supposing that$G^i = P\eta^i$, it immediately results that $D^{jk} = -\frac{1}{2}(\theta^{*j}\eta^k - \theta^{*k}\eta^j)$. Conversely, the fact that $D^{jk} = -\frac{1}{2}(\theta^{*j}\eta^k - \theta^{*k}\eta^j)$ implies $G^j\eta^k - G^k\eta^j = 0$, for which contraction with η_k leads to

$$G^i = \frac{1}{L}G^k\eta_k\eta^i = \frac{1}{2L}g^{\bar{r}k}\frac{\partial g_{j\bar{r}}}{\partial z^h}\eta^j\eta^h\eta_k\eta^i = \frac{1}{2L}\frac{\partial g_{j\bar{r}}}{\partial z^h}\eta^j\eta^h\bar{\eta}^r\eta^i = P\eta^i,$$

where $P = \frac{1}{2L}\frac{\partial g_{j\bar{r}}}{\partial z^h}\eta^j\eta^h\bar{\eta}^r$ is a smooth function on $T'M$.

A trivial computation shows that D^{jk} are $(2,1)$-homogeneous. Since $G^i = \frac{1}{2}g^{\bar{m}i}\frac{\partial g_{r\bar{m}}}{\partial z^k}\eta^k\eta^r$, we write $\frac{1}{2}g^{\bar{m}i}\frac{\partial g_{r\bar{m}}}{\partial z^k}\eta^k\eta^r = P\eta^i$. Contraction with η_i, leads to $P = \frac{1}{2L}\frac{\partial L}{\partial z^k}\eta^k$. Further on, the equality $g^{\bar{m}i}\frac{\partial g_{r\bar{m}}}{\partial z^k}\eta^k\eta^r = \frac{1}{L}\frac{\partial L}{\partial z^k}\eta^k\eta^i$ implies

$$\begin{aligned}\dot{\partial}_{\bar{r}}G^i &= -\frac{1}{2L^2}\frac{\partial L}{\partial z^k}\eta^k\eta^i\bar{\eta}_r + \frac{1}{2L}\frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^r}\eta^k\eta^i = -\frac{1}{L}G^i\bar{\eta}_r + \frac{1}{2L}\frac{\partial g_{j\bar{r}}}{\partial z^k}\eta^j\eta^k\eta^i \\ &= -\frac{1}{L}G^i\bar{\eta}_r + \frac{1}{L}G^i\bar{\eta}_r = 0,\end{aligned}$$

that is, F is generalized Berwald. □

Corollary 4.4.5. *Let (M,F) be a complex Finsler space. If D^{jk}_{hrm} and $D^{jk}_{h\bar{l}m}$ depend only on z and $\bar{z}$ and either $D^{jk} = -\frac{1}{2}(\theta^{*j}\eta^k - \theta^{*k}\eta^j)$ or $G^i = P\eta^i$, where P is a smooth function on $T'M$, then the space is Douglas.*

Proof. It results by Theorem 4.4.2 and Lemma 4.4.4. □

Lemma 4.4.6. *Let (M,F) be a complex Finsler space. Then, the functions D^{jk} are holomorphic with respect to η if and only if they are homogeneous polynomials in η of degree three and in $\bar{\eta}$ of degree zero. Moreover, given any of them, the space (M,F) is weakly Kähler.*

Proof. If D^{jk} are holomorphic with respect to η, then $D^{jk}_{\bar{r}} = 0$ and by (4.54), we have $D^{jk}_r\eta^r = 3D^{jk}$ and $\theta^{*j}\eta^k - \theta^{*k}\eta^j = 0$. Therefore, $D^{jk} = G^j\eta^k - G^k\eta^j$ which are $(3,0)$-homogeneous with respect to η. In addition, we have $D^{jk}_{\bar{r}hlm} = 0$ and owing to (4.55), $D^{jk}_{rhlm}\eta^r = 0$. The last two conditions mean that D^{jk}_{rhl} are holomorphic with respect to η and $(0,0)$-homogeneous. Thus, by the strong maximum principle, the functions D^{jk}_{rhl} depend only on z and $\bar{z}$. Taking into account (4.55), we then obtain $D^{jk} = 6D^{jk}_{rhl}(z)\eta^r\eta^h\eta^l$, which proves our claim.

Conversely, if D^{jk} are homogeneous polynomials in η of degree three and in $\bar{\eta}$ of degree zero, then there exists the functions $f^{jk}_{rhl}(z,\bar{z},\bar{\eta})$ such that $D^{jk} = f^{jk}_{rhl}(z,\bar{z},\bar{\eta})\eta^r\eta^h\eta^l$, with $f^{jk}_{rhl} = f^{jk}_{hrl} = f^{jk}_{hlr}$. Hence, $D^{jk}_{hlm} = 6f^{jk}_{hlm}(z,\bar{z},\bar{\eta})$ and $D^{jk}_{rhlm} = 0$, and from (4.55) it follows that $D^{jk}_{\bar{r}hlm}\bar{\eta}^r = 0$. By conjugation, $D^{\bar{j}\bar{k}}_{\bar{r}\bar{h}\bar{l}\bar{m}} = 0$ and $D^{\bar{j}\bar{k}}_{r\bar{h}\bar{l}\bar{m}}\eta^r = 0$ which means that the functions $D^{\bar{j}\bar{k}}_{\bar{h}\bar{l}\bar{m}}$ are holomorphic with respect to η and $(0,0)$-homogeneous. Applying again the strong maximum principle, $D^{\bar{j}\bar{k}}_{\bar{h}\bar{l}\bar{m}}$ does not depend on η or $\bar{\eta}$. Therefore, their conjugation D^{jk}_{hlm} depend only on z and $\bar{z}$ which implies $D^{jk} = \frac{1}{6}D^{jk}_{rhl}(z,\bar{z})\eta^r\eta^h\eta^l$ and thus $D^{jk}_{\bar{r}} = 0$.

Finally, after contraction on $\theta^{*j}\eta^k - \theta^{*k}\eta^j = 0$ with η_k we deduce $\theta^{*j} = 0$, that is, (M,F) is weakly Kähler. □

Theorem 4.4.7. *Let (M,F) be a complex Finsler space. The functions D^{jk} are holomorphic with respect to η if and only if (M,F) is a Kähler-Berwald space.*

Proof. If D^{jk} are holomorphic with respect to η, then $\theta^{*j} = 0$ and $D^{jk} = G^j\eta^k - G^k\eta^j$. Thus, (4.56) becomes

$$\begin{aligned}D^{jk}_{hlrm} &= (\dot{\partial}_m G^j_{hlr})\eta^k + G^j_{hlr}\delta^k_m + G^j_{hlm}\delta^k_r + G^j_{hmr}\delta^k_l + G^j_{mlr}\delta^k_h - [j,k], \\ D^{jk}_{h\bar{l}\bar{r}m} &= (\dot{\partial}_m G^j_{h\bar{l}\bar{r}})\eta^k + G^j_{h\bar{l}\bar{r}}\delta^k_m + G^j_{m\bar{l}\bar{r}}\delta^k_h - [j,k], \\ D^{jk}_{h\bar{l}rm} &= (\dot{\partial}_m G^j_{h\bar{l}r})\eta^k + G^j_{h\bar{l}r}\delta^k_m + G^j_{h\bar{l}m}\delta^k_r + G^j_{m\bar{l}r}\delta^k_h - [j,k].\end{aligned}$$

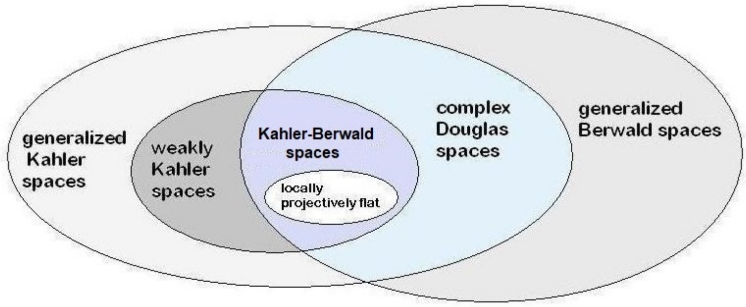


Figure 4.1: Inclusion diagram

Setting $k = m$ and using the formulas (4.2) and (4.4), these yield

$$\begin{aligned} D^{jk}_{hlrk} &= (n+1)\{G^j_{hlr} - \frac{1}{n+1}[(\dot{\partial}_r D_{hl})\eta^j + \sum_{(h,l,r)} D_{hl}\delta^j_r]\}, \\ D^{jk}_{h\bar{l}\bar{r}k} &= (n+1)\{G^j_{h\bar{l}\bar{r}} - \frac{1}{n+1}[(\dot{\partial}_h D_{\bar{l}\bar{r}})\eta^j + D_{\bar{l}\bar{r}}\delta^j_h]\}, \\ D^{jk}_{h\bar{l}rk} &= (n+1)\{G^j_{h\bar{l}r} - \frac{1}{n+1}[(\dot{\partial}_r D_{\bar{l}h})\eta^j + D_{\bar{l}h}\delta^j_r + D_{\bar{l}r}\delta^j_h]\}. \end{aligned}$$

Substituting (4.15) with $\theta^{*j} = 0$ into these, we obtain $D^{jk}_{hlrk} = (n+1)D^j_{hlr}$, $D^{jk}_{h\bar{l}\bar{r}k} = (n+1)D^j_{h\bar{l}\bar{r}}$ and $D^{jk}_{h\bar{l}rk} = (n+1)D^j_{h\bar{l}r}$. However, we also have $D^{jk}_{rhlm} = D^{jk}_{h\bar{l}rm} = D^{jk}_{h\bar{l}\bar{r}m} = 0$. Thus, $D^j_{hlr} = D^j_{h\bar{l}\bar{r}} = D^j_{h\bar{l}r} = 0$, that is, the space is Douglas. Moreover, by Theorem 4.2.17 it is Kähler-Berwald.

Conversely, if the space is Kähler-Berwald, then $\theta^{*j} = 0$ and $\dot{\partial}_{\bar{r}} G^i = 0$. These lead to $D^{jk} = G^j\eta^k - G^k\eta^j$ and $D^{jk}_{\bar{r}} = 0$. □

Using Lemma 4.4.6 and Theorem 4.4.7, it is justified the following result.

Corollary 4.4.8. *Let (M, F) be a complex Finsler space. Then, the functions D^{jk} are homogeneous polynomials in η of degree three and in $\bar{\eta}$ of degree zero if and only if (M, F) is a Kähler-Berwald space.*

Consider F a complex Finsler metric on the complex manifold M which is locally projectively flat. Thus, as we already proved, the spray coefficients corresponding to F are $G^i = \frac{1}{2L}\frac{\partial L}{\partial z^k}\eta^k\eta^i$ and F is weakly Kähler (i.e. $\theta^{*i} = 0$, see Section 4.3 and [24]), which give $D^{jk} = 0$. Moreover, any locally projectively flat complex Finsler metric is a complex Kähler-Berwald metric. The converse is not true, but any complex Kähler-Berwald metric with $G^i = P\eta^i$, where P is a smooth function on $T'M$, is a locally projectively flat metric.

Combining all the results proved above, we can present the inclusion diagram shown in Figure 4.1. In the next section we describe an interesting family of complex Douglas spaces.

4.5 Complex Douglas spaces with Randers metrics

Let $\tilde{a} = a_{i\bar{j}}(z)dz^i \otimes d\bar{z}^j$ be a pure Hermitian metric and let $b = b_i(z)dz^i$ be a differential $(1,0)$-form, both on M. By these objects we have defined (for more details see [27, 26, 25]) the complex Randers metric $F(z,\eta)=\alpha+|\beta|$, where $\alpha(z,\eta)=\sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$ and $\beta(z,\eta)=b_i(z)\eta^i$.

We note that the complex Randers metrics are arresting in complex Finsler geometry, and they represent a medium such that Hermitian geometry interferes with complex Finsler geometry properly. Since any pure Hermitian metric is a complex Douglas metric, our next study is focused on the complex Randers metrics with $\beta \neq 0$. Recall that for a complex Randers metric we have

$$\begin{aligned}
\frac{\partial\alpha}{\partial\eta^i} &= \frac{1}{2\alpha}l_i, \quad \frac{\partial|\beta|}{\partial\eta^i}=\frac{\bar{\beta}}{2|\beta|}b_i, \quad \eta_i=\frac{\partial L}{\partial\eta^i}=\frac{F}{\alpha}l_i+\frac{F\bar{\beta}}{|\beta|}b_i, \\
g_{i\bar{j}} &= \frac{F}{\alpha}a_{i\bar{j}}-\frac{F}{2\alpha^3}l_il_{\bar{j}}+\frac{F}{2|\beta|}b_ib_{\bar{j}}+\frac{1}{2L}\eta_i\eta_{\bar{j}}, \\
g^{\bar{j}i} &= \frac{\alpha}{F}a^{\bar{j}i}+\frac{|\beta|(\alpha||b||^2+|\beta|)}{L\gamma}\eta^i\bar{\eta}^j-\frac{\alpha^3}{F\gamma}b^i\bar{b}^j-\frac{\alpha}{F\gamma}(\bar{\beta}\eta^i\bar{b}^j+\beta b^i\bar{\eta}^j), \\
N^i_j &= \overset{a}{N^i_j}+\frac{1}{\gamma}(l_{\bar{r}}\frac{\partial\bar{b}^r}{\partial z^j}-\frac{\beta^2}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\xi^i+\frac{\beta}{2|\beta|}k^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j},
\end{aligned} \tag{4.59}$$

where $k^{\bar{r}i}=2\alpha a^{\bar{r}i}+\frac{2(\alpha||b||^2+2|\beta|)}{\gamma}\eta^i\bar{\eta}^r-\frac{2\alpha^3}{\gamma}b^i\bar{b}^r-\frac{2\alpha}{\gamma}(\bar{\beta}\eta^i\bar{b}^r+\beta b^i\bar{\eta}^r)$, $\gamma=L+\alpha^2(||b||^2-1)$, $\xi^i=\bar{\beta}\eta^i+\alpha^2b^i$, $\overset{a}{N^k_j}=a^{\bar{m}k}\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l$, with the settings $b^i=a^{\bar{j}i}b_{\bar{j}}$, $||b||^2=a^{\bar{j}i}b_ib_{\bar{j}}$, $b^{\bar{i}}=\bar{b}^i$. Therefore, the spray coefficients are

$$G^i=\overset{a}{G^i}+\frac{1}{2\gamma}(l_{\bar{r}}\frac{\partial\bar{b}^r}{\partial z^j}-\frac{\beta^2}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\xi^i\eta^j+\frac{\beta}{4|\beta|}k^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j. \tag{4.60}$$

and for the generalized Berwald Randers spaces we have proven Theorem 2.3.5 which attest that a connected complex Randers space (M,F) is a generalized Berwald space if and only if $(\bar{\beta}l_{\bar{r}}\frac{\partial\bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=0$. Moreover, one has that $G^i=\overset{a}{G^i}$.

All subsequent reasoning is under assumptions of generalized Berwald property. Since $G^i=\overset{a}{G^i}$, then $\overset{c}{N^k_j}=\frac{1}{2}a^{\bar{m}k}(\frac{\partial a_{l\bar{m}}}{\partial z^j}+\frac{\partial a_{j\bar{m}}}{\partial zl})\eta^l$, which together with (4.59), leads to

$$\overset{c}{\delta}_{\bar{m}}L=-\frac{F}{2}\Gamma_{l\bar{r}\bar{m}}\tau^l\bar{\eta}^r-\frac{F\beta}{|\beta|}\Omega_{\bar{m}}, \tag{4.61}$$

where $\Gamma_{l\bar{r}\bar{m}}=\frac{\partial a_{l\bar{m}}}{\partial\bar{z}^r}-\frac{\partial a_{l\bar{r}}}{\partial\bar{z}^m}$, $\Omega_{\bar{m}}=\overset{a}{N^{\bar{s}}_{\bar{m}}}b_{\bar{s}}-\frac{\partial b_{\bar{r}}}{\partial\bar{z}^m}\bar{\eta}^r-\frac{\bar{\beta}^2}{|\beta|^2}\frac{\partial b_l}{\partial\bar{z}^m}\eta^l$, $\tau^l=\frac{1}{F}a^{\bar{m}l}\bar{\eta}_m=\frac{1}{\alpha}\eta^l+\frac{\beta}{|\beta|}b^l$.

From (4.61) and (4.59), it follows that

$$\begin{aligned}
\overset{a}{\theta^{*i}} &= -\Gamma_{l\bar{r}\bar{m}}a^{\bar{m}i}\eta^l\bar{\eta}^r, \\
\theta^{*i} &= -\alpha(\Gamma_{l\bar{r}\bar{m}}\tau^l\bar{\eta}^r+\frac{2\beta}{|\beta|}\Omega_{\bar{m}})(h^{\bar{m}i}-\frac{\bar{\beta}}{\gamma}b^{\bar{m}}\eta^i),
\end{aligned} \tag{4.62}$$

where $h^{\bar{m}i}=a^{\bar{m}i}-\frac{\alpha^2}{\gamma}b^{\bar{m}}b^i$.

Once we obtain θ^{*i} and $\overset{a}{\theta^{*i}}$, a technical computation yields the expressions for K^i and $\overset{a}{K^i}$. This involves some trivial calculus, which leads to

$$\begin{aligned}
&\overset{a}{K^i}=-\Gamma_{l\bar{r}\bar{m}}(a^{\bar{m}i}\eta^l-\tfrac{1}{n}a^{\bar{m}l}\eta^i)\bar{\eta}^r, \\
&K^i=-\Gamma_{l\bar{r}\bar{m}}[\alpha h^{\bar{m}i}\tau^l-\tfrac{1}{n}(a^{\bar{m}l}-\tfrac{\omega\alpha\bar{\beta}}{\gamma}b^{\bar{m}}\tau^l)\eta^i]\bar{\eta}^r-2\Omega_{\bar{m}}[\tfrac{\alpha\beta}{|\beta|}h^{\bar{m}i}+\tfrac{\alpha|\beta|}{n\gamma}(1-\tfrac{\alpha}{|\beta|}-\tfrac{\alpha^2\omega}{\gamma})b^{\bar{m}}\eta^i],
\end{aligned} \tag{4.63}$$

where $\omega=1-\frac{\alpha}{|\beta|}+\frac{2\alpha^2(1-||b||^2)}{\gamma}$.

Theorem 4.5.1. *Let (M,F) be a connected complex Randers space. (M,F) is a complex Douglas space if and only if $(\bar{\beta}l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=0$ and $K^i=\overset{a}{K^i}$. Given any of them, $\Omega_{\bar{m}}=-\frac{1}{2}\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$. Moreover, if α is Kähler, then (M,F) is a Kähler-Berwald space.*

Proof. If (M,F) is complex Douglas space, then $(\bar{\beta}l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=0$ and $K^i=\varphi^i_{\bar{r}s}(z,\bar{z})\bar{\eta}^r\eta^s$, which means that K^i are homogeneous polynomials in η and in $\bar{\eta}$ of first degree. Thus, using (4.63) we have

$$\begin{aligned}&\alpha|\beta|\Big\{K^i+\Gamma_{l\bar{r}\bar{m}}\Big[h^{\bar{m}i}\bar{\eta}^r\eta^l-\tfrac{1}{n}[a^{\bar{m}l}-\tfrac{\bar{\beta}}{\gamma}(1-\tfrac{2\alpha^2||b||^2}{\gamma})b^{\bar{m}}\eta^l+\tfrac{\alpha^2(\alpha^2||b||^2+|\beta|^2)}{\gamma}b^{\bar{m}}b^l)]\bar{\eta}^r\eta^i\Big]\\&+\tfrac{2\alpha^2}{n\gamma}(1+\tfrac{\alpha^2||b||^2+|\beta|^2}{\gamma})\Omega_{\bar{m}}b^{\bar{m}}\eta^i\Big\}+\Gamma_{l\bar{r}\bar{m}}\Big[\alpha^2\beta h^{\bar{m}i}\bar{\eta}^rb^l-\tfrac{1}{n\gamma}[\tfrac{\alpha^2\bar{\beta}(\alpha^2||b||^2+|\beta|^2)}{\gamma}b^{\bar{m}}\eta^l\\&-\alpha^2|\beta|^2(1-\tfrac{2\alpha^2||b||^2}{\gamma})b^{\bar{m}}b^l]\bar{\eta}^r\eta^i\Big]+2\Omega_{\bar{m}}\Big[\alpha^2\beta h^{\bar{m}i}+\tfrac{\alpha^2|\beta|^2}{n\gamma}(1-\tfrac{2\alpha^2||b||^2}{\gamma})b^{\bar{m}}\eta^i\Big]=0,\end{aligned}$$

which contains an irrational part and a rational part. We can deduce that

$$\begin{aligned}&K^i=-\Gamma_{l\bar{r}\bar{m}}\{h^{\bar{m}i}\bar{\eta}^r\eta^l-\tfrac{1}{n}[a^{\bar{m}l}-\tfrac{\bar{\beta}}{\gamma}(1-\tfrac{2\alpha^2||b||^2}{\gamma})b^{\bar{m}}\eta^l+\tfrac{\alpha^2(\alpha^2||b||^2+|\beta|^2)}{\gamma}b^{\bar{m}}b^l)]\bar{\eta}^r\eta^i\}\\&\quad-\tfrac{2\alpha^2}{n\gamma}(1+\tfrac{\alpha^2||b||^2+|\beta|^2}{\gamma})\Omega_{\bar{m}}b^{\bar{m}}\eta^i \text{ and}\\&\Gamma_{l\bar{r}\bar{m}}\{\alpha^2\beta h^{\bar{m}i}\bar{\eta}^rb^l-\tfrac{1}{n\gamma}[\tfrac{\alpha^2\bar{\beta}(\alpha^2||b||^2+|\beta|^2)}{\gamma}b^{\bar{m}}\eta^l-\alpha^2|\beta|^2(1-\tfrac{2\alpha^2||b||^2}{\gamma})b^{\bar{m}}b^l]\bar{\eta}^r\eta^i\}\\&\quad=-2\Omega_{\bar{m}}\{\alpha^2\beta h^{\bar{m}i}+\tfrac{\alpha^2|\beta|^2}{n\gamma}(1-\tfrac{2\alpha^2||b||^2}{\gamma})b^{\bar{m}}\eta^i\}.\end{aligned}\tag{4.64}$$

Contractions with l_i and b_i in the second formula in (4.64) yield the homogeneous linear system

$$\left\{\begin{array}{c}|\beta|^2(n-1+\frac{2\alpha^2||b||^2}{\gamma})X+\frac{\bar{\beta}(\alpha^2||b||^2+|\beta|^2)}{\gamma}Y=0\\ [n(\gamma-\alpha^2||b||^2)+|\beta|^2(1-\frac{2\alpha^2||b||^2}{\gamma})]X-\frac{\bar{\beta}(\alpha^2||b||^2+|\beta|^2)}{\gamma}Y=0\end{array}\right.\tag{4.65}$$

with the unknowns $X=(\Gamma_{l\bar{r}\bar{m}}\bar{\eta}^rb^l+2\Omega_{\bar{m}})b^{\bar{m}}$ and $Y=\Gamma_{l\bar{r}\bar{m}}\eta^l\bar{\eta}^rb^{\bar{m}}$. Since its determinant is nonzero, $\Delta=-\frac{2n\bar{\beta}|\beta|F(\alpha^2||b||^2+|\beta|^2)}{\gamma}\neq 0$, then the system (4.65) admits only the null solution, that is,

$$\begin{aligned}(\Gamma_{l\bar{r}\bar{m}}\bar{\eta}^rb^l+2\Omega_{\bar{m}})b^{\bar{m}}&=0,\\ \Gamma_{l\bar{r}\bar{m}}\eta^l\bar{\eta}^rb^{\bar{m}}&=0.\end{aligned}\tag{4.66}$$

By derivations with respect to η and $\bar{\eta}$, the second relation in (4.66) implies $\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}}=0$. When substituted in the first relation in (4.66), this yields $\Omega_{\bar{m}}b^{\bar{m}}=0$. These, along with (4.64), lead to $K^i=\overset{a}{K^i}$.

Conversely, since $\overset{a}{K^i}$ are always homogeneous polynomials in η and in $\bar{\eta}$ of first degree and $K^i=\overset{a}{K^i}$, then (M,F) is a generalized Kähler space. This together with Theorem 2.3.5 completes our claim.

If the space is complex Douglas, then $\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}}=\Omega_{\bar{m}}b^{\bar{m}}=0$. When substituted into the second formula in (4.64), this yields $\Omega_{\bar{m}}=-\frac{1}{2}\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$.

Moreover, if α is Kähler, then $K^i=\overset{a}{K^i}=0$. According to Theorem 4.2.17, (M,F) is a Kähler-Berwald space. □

Theorem 4.5.2. *Let (M,F) be a connected complex Randers space. If (M,F) is a generalized Berwald space then, $K^i=\overset{a}{K^i}$ if and only if $\theta^{*i}=\overset{a}{\theta^{*i}}$.*

Proof. Under the assumption of generalized Berwald property, we have the relations (4.62) and (4.63). If $K^i = \overset{a}{K^i}$ then, the second formula in (4.63) can be rewritten as the sum of an irrational part and a rational part. This implies that

$$\{-\tfrac{\alpha^2}{\gamma}\eta^l b^i + \tfrac{1}{n}[\tfrac{\bar{\beta}}{\gamma}(1 - \tfrac{2\alpha^2||b||^2}{\gamma})\eta^l - \tfrac{\alpha^2(\alpha^2||b||^2+|\beta|^2)}{\gamma}b^l)]\eta^i\}\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}}\bar{\eta}^r$$

$$+ \tfrac{2\alpha^2}{n\gamma}(1 + \tfrac{\alpha^2||b||^2+|\beta|^2}{\gamma})\Omega_{\bar{m}}b^{\bar{m}}\eta^i = 0$$

and the second formula (4.64). Similar reasoning to that in the proof of Theorem 4.5.1, gives $\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}} = 0$ and $\Omega_{\bar{m}}b^{\bar{m}} = 0$. Therefore, $\Omega_{\bar{m}} = -\frac{1}{2}\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$. Substitution of the last three relations in (4.62), leads to $\theta^{*i} = \overset{a}{\theta^{*i}}$.

Conversely, if $\theta^{*i} = \overset{a}{\theta^{*i}}$ then, the second formula (4.62) is

$$-\frac{\alpha|\beta|}{\gamma}\Gamma_{l\bar{r}\bar{m}}\eta^l\bar{\eta}^r(\alpha^2 b^i + \bar{\beta}\eta^i)b^{\bar{m}} + \alpha^2\beta(\Gamma_{l\bar{r}\bar{m}}\bar{\eta}^r b^l + 2\Omega_{\bar{m}})(h^{\bar{m}i} - \frac{\bar{\beta}}{\gamma}b^{\bar{m}}\eta^i) = 0,$$

which contains an irrational part and a rational part. It follows that

$$\begin{aligned} \Gamma_{l\bar{r}\bar{m}}\eta^l\bar{\eta}^r(\alpha^2 b^i + \bar{\beta}\eta^i)b^{\bar{m}} &= 0, \\ (\Gamma_{l\bar{r}\bar{m}}\bar{\eta}^r b^l + 2\Omega_{\bar{m}})(h^{\bar{m}i} - \frac{\bar{\beta}}{\gamma}b^{\bar{m}}\eta^i) &= 0. \end{aligned} \tag{4.67}$$

Contracting in (4.67) with either l_i or b_i, we obtain $\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}} = 0$, $\Omega_{\bar{m}}b^{\bar{m}} = 0$ and $\Omega_{\bar{m}} = -\frac{1}{2}\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$. Substituted in (4.63), these give $K^i = \overset{a}{K^i}$. □

An immediate consequence of above theorems follows.

Theorem 4.5.3. *Let (M, F) be a connected complex Randers space. (M, F) is a complex Douglas space if and only if $(\bar{\beta}l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j} + \beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j = 0$ and $\theta^{*i} = \overset{a}{\theta^{*i}}$.*

To establish another characteristic of complex Douglas spaces with Randers metric, we recall Theorem 3.3.2 i).

Theorem 4.5.4. *Let (M, F) be a connected complex Randers space. Then, α and F are projectively related if and only if F is generalized Berwald and $B^i = -P\eta^i$, for any $i = \overline{1, n}$, where $P = -\frac{\bar{\beta}}{4F|\beta|}\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}}\eta^l\bar{\eta}^r$.*

Theorem 4.5.5. *Let (M, F) be a connected complex Randers space. (M, F) is a complex Douglas space if and only if α and F are projectively related.*

Proof. If (M, F) is a complex Douglas space, then by Theorem 4.5.3, we have $\theta^{*i} = \overset{a}{\theta^{*i}}$ and thus, $B^i = 0$. Moreover, by Theorem 2.3.5, we have $G^i = \overset{a}{G^i}$. Hence $P = 0$ and, according to Theorem 4.5.4, the metrics α and F are projectively related. Conversely, if α and F are projectively related and since α is a complex Douglas metric, according to Theorem 4.2.8, the Randers metric F is also a complex Douglas metric. □

Now we consider a connected complex Randers metric $F=\alpha+|\beta|$, on a two-dimensional complex manifold M. Assuming that $F=\alpha+|\beta|$ is a complex Douglas metric, it satisfies the conditions

$$\Gamma_{l\bar{r}\bar{1}}b^{\bar{1}}+\Gamma_{l\bar{r}\bar{2}}b^{\bar{2}}=0, \quad \text{with } l,r=1,2. \tag{4.68}$$

Since $\Gamma_{l\bar{m}\bar{m}}=0$ and $\Gamma_{l\bar{1}\bar{2}}=-\Gamma_{l\bar{2}\bar{1}}$, with $l,m=1,2$, (4.68) reduces to $\Gamma_{l\bar{1}\bar{2}}b^{\bar{1}}=0$ and $\Gamma_{l\bar{1}\bar{2}}b^{\bar{2}}=0$. These give $\Gamma_{l\bar{1}\bar{2}}=0$, because at least one of coefficients $b^{\bar{m}}$ is nonzero. This means that the metric α is Kähler, and thus, by Theorem 4.5.1, a complex Randers Douglas space of dimension two is Kähler-Berwald space.

However, there exist the complex Randers Douglas spaces that are not Kähler-Berwald for dimension $n\geq 3$. We show this fact using an example. Hereinafter, we construct an explicit example of complex Randers metric that is complex Douglas, i.e. it satisfies the conditions of Theorems 4.5.1 and 4.5.3.

On $M=\mathbf{C}^3$ we set the pure Hermitian metric

$$\alpha^2=e^{z^1+\bar{z}^1}\left|\eta^1\right|^2+e^{z^2+\bar{z}^2}\left|\eta^2\right|^2+e^{z^1+\bar{z}^1+z^3+\bar{z}^3}\left|\eta^3\right|^2 \tag{4.69}$$

and we choose the $(1,0)$-differential form β given by$\beta=e^{z^2}\eta^2$. Then, $|\beta|^2=e^{z^2+\bar{z}^2}\left|\eta^2\right|^2$ and thus $b_i=b^i=0$, $i\in\{1,3\}$, $b_2=e^{z^2}$ and $b^2=e^{-z^2}$. In addition, we have $\Gamma_{l\bar{r}\bar{m}}=0$, excepting the coefficients $\Gamma_{3\bar{1}\bar{3}}=-\Gamma_{3\bar{3}\bar{1}}=e^{z^1+\bar{z}^1+z^3+\bar{z}^3}\neq 0$. Thus, the metric (4.69) is not Kähler. With these tools we construct the complex Randers metric

$$F=\sqrt{e^{z^1+\bar{z}^1}\left|\eta^1\right|^2+e^{z^2+\bar{z}^2}\left|\eta^2\right|^2+e^{z^1+\bar{z}^1+z^3+\bar{z}^3}\left|\eta^3\right|^2}+\sqrt{e^{z^2+\bar{z}^2}\left|\eta^2\right|^2}, \tag{4.70}$$

which has $\det(g_{i\bar{j}})=\frac{F^5}{2\alpha^4|\beta|}\det(a_{i\bar{j}})>0$, $i,j=1,2,3$. Some computations lead to the conclusion that the the metric (4.70) is generalized Berwald, this is,

$$(\bar{\beta}l_{\bar{r}}\frac{\partial\bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=(\bar{\beta}l_{\bar{2}}\frac{\partial\bar{b}^2}{\partial z^j}+\beta\frac{\partial b_{\bar{2}}}{\partial z^j}\bar{\eta}^2)\eta^j=0.$$

Moreover, we have $\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}}=\Gamma_{l\bar{r}\bar{2}}b^{\bar{2}}=0$, $\underset{a}{\Gamma_{l\bar{r}\bar{m}}}b^l=\underset{a}{\Gamma_{2\bar{r}\bar{m}}}b^2=0$, $\underset{a}{\Omega_{\bar{r}}}=0$, $r=1,2,3$. If we substitute these into (4.62) and (4.63), it turns out $\theta^{*i}=\theta^{*i}$ and $K^i=K^i$, $i=1,2,3$. Thus, by Theorems 4.5.1 and 4.5.3, (4.70) is a complex Douglas metric.

We note that the above example can be generalized to examples of complex Douglas metric, taking on $M=\mathbf{C}^n$, $\alpha^2=\sum\limits_{k=1,\,k\neq 3}^{n}e^{z^k+\bar{z}^k}\left|\eta^k\right|^2+e^{z^1+\bar{z}^1+z^3+\bar{z}^3}\left|\eta^3\right|^2$. For β we can choose one of the following possibilities $\beta=e^{z^k}\eta^k$, where $k=\overline{1,n}$, excepting $k=1$ and 3.

Chapter 5

Zermelo's deformation of Hermitian metrics

In this chapter, mainly based on the paper [9], we first present Zermelo's navigation on Hermitian manifolds making use of the real homogeneous complex Finsler metrics (briefly $\mathbb{R}$-complex Finsler metrics). More precisely, $\mathbb{R}$-complex Randers metrics are obtained by Zermelo's deformation of the Hermitian metrics, with space-dependent ship's relative speed under the action of weak complex vector fields. Next, we indicate the behaviour of certain properties of a Hermitian metric under Zermelo deformation in a special holomorphic wind.

5.1 Motivation and the main results

In Zermelo's navigation problem, formulated initially by E. Zermelo in [157], the objective is to find the paths which minimize travel time of the ship proceeding from one point to another point in the presence of perturbing wind W, under assumption the ship sails at constant maximum speed relative to the surrounding sea. Exploration of this problem has led to important generalizations and results in Riemann-Finsler geometry. It was shown that the solutions of Zermelo's problem on Riemannian manifold (M, h) are represented by geodesics of a Randers metric (in weak wind) or Kropina metric (in critical wind) [45, 124, 76, 154, 93, 96]. This subject is still being addressed because of its various applications in the essential theoretical investigations [45, 124, 76, 84, 154, 93, 96] as well as in the real world problems [104, 95, 94].

Zermelo's navigation was also considered by us on Hermitian manifolds, where the solutions are represented by the complex Randers and complex Kropina metrics [40, 14, 15]. It is worth mentioning that, in contrast to a real analogue, a complex Randers metric commenced to be studied much later (2007, cf. [36]). In order to benefit from similar interest like this given by real Randers metrics, it was natural to find some applications for them. Thus, the concepts in real setting presented in [45] were referred by us in [40], where the navigation problem was investigated on a Hermitian manifold. The application of a navigation representation in a complex landscape enabled to obtain the concrete examples of complex Randers metrics [40] and to point out the essential difference in comparison to the analogous problem on Riemannian manifolds. Namely, the complex Randers metrics are not of constant holomorphic curvature by perturbation of some Hermitian metrics of constant holomorphic sectional curvature via the Zermelo's navigation [40]. Nevertheless, it was necessary to work out the additional geometric assumptions which come from the complex homogeneity requirement for complex Finsler metrics [14, 15]. Without this restriction, the solutions of the problem are only real homogeneous, actually $\mathbb{R}$-complex Finsler metrics. These have been developed in [120, 41, 21, 22]. Thus, the first purpose of this chapter is to describe $\mathbb{R}$-complex Finsler metrics as solutions of Zermelo's navigation problem on Hermitian manifolds (M, h), under action of weak winds W and with variable space-dependent ship's relative speed $||u||_h$. The second objective is to investigate the holomorphic curvature of a class of $\mathbb{R}$-complex Finsler metrics obtained by Zermelo's navigation, that is, as the deformation of some Hermitian metrics by certain holomorphic vector fields.

An overview of the chapter's content. In Section 5.2 we summarize some preliminary notions on n-dimensional $\mathbb{R}$-complex Finsler spaces. In Section 5.3, following [93], we describe the generalized Zermelo navigation problem (briefly, generalized ZNP) on Hermitian manifolds, the main result being Theorem 5.3.3, which attests that the $\mathbb{R}$-complex Hermitian Finsler metrics are of Randers type if and only if they solve the generalized ZNP on Hermitian manifolds. Besides the meaning that the generalized ZNP provides a concrete application for the $\mathbb{R}$-complex Hermitian Randers metrics, more valuable is the fact that by generalized ZNP we can construct explicit non-Hermitian metrics (called W-Zermelo deformations), deforming the background Hermitian metric h by given data W and $||u(z)||_h$. This is followed in Section 5.4, where we study the holomorphic curvature of some W-Zermelo deformations F, taking into consideration holomorphic vector fields. Assuming that the holomorphic curvature of W-Zermelo deformation F is only space-dependent and W is a special holomorphic wind, we prove that the holomorphic curvature of F is vanishing (Theorem 5.4.8). Moreover, we study the effects of the Zermelo deformation on some properties of a Hermitian metric h, e.g. Kähler property and the holomorphic sectional curvature (Theorems 5.4.6, 5.4.7 and Corollary 5.4.9).

5.2 Few rudiments of $\mathbb{R}$-complex Finsler geometry

To begin with, we point out some basic notions from $\mathbb{R}$-complex Finsler geometry [1, 116, 120, 41, 21, 22, 26]. Next, we introduce the class of $\mathbb{R}$-complex Randers metrics which solve the Zermelo's navigation problem in the Hermitian landscape. This is more general approach to the subject than the class studied in [41, 22].

Basic notions and notations. Let M be an n-dimensional complex manifold and $(z^k)_{k=\overline{1,n}}$ be the complex coordinates in a local chart in $z \in M$. We recall that $T'M$ denotes the holomorphic tangent bundle, $T'M$ being a $2n$-dimensional complex manifold with $(z^k, \eta^k)_{k=\overline{1,n}}$ the local coordinates in $(z, \eta) \in T'M$.

Definition 5.2.1. *[120] An $\mathbb{R}$-complex Finsler space is a pair (M, F), where F is a continuous function $F : T'M \longrightarrow \mathbb{R}_+$ satisfying the conditions:*
i) $L = F^2$ is smooth on $\widetilde{M} = T'M \backslash \{0\}$;
ii) $F(z, \eta) \geq 0$ for all $(z, \eta) \in T'M$; the equality holds if and only if $\eta = 0$;
iii) $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = \lambda F(z, \eta, \bar{z}, \bar{\eta})$, for all $\lambda > 0$.

We note that this definition refers to complex Finsler metrics, where we restrict the homogeneity condition *iii)* to the real scalars. The Hessian and the Levi matrices of L induce the tensors

$$g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}, \quad g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}, \quad g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \bar{\eta}^j}, \tag{5.1}$$

that obey the below properties

$$\begin{aligned} &(\dot{\partial}_i L)\eta^i + (\dot{\partial}_{\bar{i}} L)\bar{\eta}^i = 2L, \qquad g_{ij}\eta^i + g_{j\bar{i}}\bar{\eta}^i = \dot{\partial}_j L, \qquad L = \mathrm{Re}\{g_{ij}\eta^i\eta^j\} + g_{i\bar{j}}\eta^i\bar{\eta}^j, \\ &(\dot{\partial}_j g_{ik})\eta^j + (\dot{\partial}_{\bar{j}} g_{ik})\bar{\eta}^j = 0, \qquad (\dot{\partial}_j g_{i\bar{k}})\eta^j + (\dot{\partial}_{\bar{j}} g_{i\bar{k}})\bar{\eta}^j = 0. \end{aligned} \tag{5.2}$$

They are obtained as consequences of the real homogeneity condition *iii)*.

Subsequently, a smooth function $\varphi(z,\eta)$ with $(\dot{\partial}_i\varphi)\eta^i + (\dot{\partial}_{\bar{i}}\varphi)\bar{\eta}^i = s\varphi$ is called $\mathbb{R}$-homogenous of degree s in the fibre variable η. Thus, according to (5.2) L is $\mathbb{R}$-homogenous of degree 2 in the fibre variable and the tensors g_{ik} and $g_{i\bar{k}}$ are $\mathbb{R}$-homogenous of degree 0.

As mentioned in the Chapter 1, a strong pseudoconvexity is assumed in complex Finsler geometry. This implies the matrix $(g_{i\bar{j}})$ is positive definite on $\widetilde{M}$. But here, due to the real homogeneity condition the fundamental function L acquires a more general form than in complex Finsler geometry, namely $L = \mathrm{Re}\{g_{ij}\eta^i\eta^j\} + g_{i\bar{j}}\eta^i\bar{\eta}^j$ by (5.2). Consequently, there are highlighted two general classes of $\mathbb{R}$-complex Finsler spaces. Namely, the class of $\mathbb{R}$-*complex Hermitian Finsler spaces*, i.e. the Levi matrix $(g_{i\bar{j}})$ is positive definite on $\widetilde{M}$, and the class of $\mathbb{R}$-*complex non-Hermitian Finsler spaces*, i.e. the Hessian matrix (g_{ij}) is positive definite on $\widetilde{M}$. Thus, in this way two different geometries were developed on $T'M$. That is, Hermitian and non-Hermitian, related by L (see [120]) and actually also the geometry of the *strongly convex* $\mathbb{R}$-complex Finsler spaces, where both matrices $(g_{i\bar{j}})$ and (g_{ij}) are positive definite on $\widetilde{M}$ (see [21]).

Further on, for our purpose we ought to focus on the class of $\mathbb{R}$-complex Hermitian Finsler spaces. In particular, if $g_{i\bar{j}}$ depends only on z, then we called an $\mathbb{R}$-complex Hermitian Finsler space with $g_{i\bar{j}}(z)$ a *pure Hermitian* space. Note that also g_{ij} depends only on z (cf. [22]) and the fundamental function L is smooth on whole $T'M$ in this case.

In $\mathbb{R}$-complex Hermitian Finsler geometry Chern-Finsler complex nonlinear connection (with the local coefficients $N^i_j = g^{\bar{m}i}\frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^m} = g^{\bar{m}i}[(\partial_k g_{\bar{r}\bar{m}})\bar{\eta}^r + (\partial_k g_{s\bar{m}})\eta^s])$ is the main tool to study [120]. Thus, from now on, δ_k is considered with respect to the above nonlinear connection.

Also, Chern-Finsler complex nonlinear connection induces a complex spray

$$S = \eta^i \frac{\partial}{\partial z^i} - 2G^i(z,\eta)\dot{\partial}_i,$$

where $2G^i = N^i_j\eta^j$. Since L is $\mathbb{R}$-homogenous of degree 2 in the fibre variables, the local coefficients N^i_j and G^i are $\mathbb{R}$-homogeneous of degree 1 and 2, respectively, with respect to the fibre variables.

Moreover, the *Chern-Finsler connection* $D : \Gamma(T'\widetilde{M}) \to \Gamma(T^*_{\mathbb{C}}\widetilde{M} \otimes T'\widetilde{M})$ (i.e. metrical compatible and of $(1,0)-$ type, see [120, 21]) is retrieved in an $\mathbb{R}$-complex Hermitian Finsler space. Locally D is given by

$$L^i_{jk} = g^{\bar{m}i}(\delta_j g_{k\bar{m}}), \quad C^i_{jk} = g^{\bar{m}i}(\dot{\partial}_j g_{k\bar{m}}), \quad L^i_{j\bar{k}} = C^i_{j\bar{k}} = 0, \tag{5.3}$$

and it has the properties $L^i_{jk} = \dot{\partial}_j N^i_k$ and $N^i_k = L^i_{jk}\eta^j + (\dot{\partial}_{\bar{r}} N^i_k)\bar{\eta}^r$ [21].

If in addition F is assumed to satisfy $T^i_{jk} = 0$ or $T^i_{jk}\eta^j = 0$, where $T^i_{jk} = L^i_{jk} - L^i_{kj}$, then the $\mathbb{R}$-complex Hermitian Finsler space (M,F) is called *strongly Kähler* or *Kähler*, respectively. Any strongly Kähler metric is a Kähler metric and, in the pure Hermitian case both these types are the same with $\frac{\partial g_{j\bar{m}}}{\partial z^i} = \frac{\partial g_{i\bar{m}}}{\partial z^j}$.

Let $\gamma:[0,1]\to M$, $\gamma(t)=(\gamma^k(t))=(z^k(t))$, $k=\overline{1,n}$ with a real parameter t and the velocity $\frac{d\gamma^k}{dt}=\eta^k(t)$, be a regular curve with $F\left(\gamma(t),\frac{d\gamma(t)}{dt}\right)=c>0$. γ is a geodesic curve corresponding to the $\mathbb{R}$-complex Hermitian Finsler function F on M iff $\frac{d}{dt}\left(\frac{\partial L}{\partial\eta^i}(\gamma,\dot\gamma)\right)=\frac{\partial L}{\partial z^i}(\gamma,\dot\gamma)$, where $L=F^2$, and $\dot\gamma=\frac{d\gamma}{dt}$ (see [120]). The last equation is equivalent to

$$\left(g_{ij}\frac{d^2\gamma^j}{dt^2}+\frac{\partial^2 L}{\partial z^j\partial\eta^i}\eta^j-\frac{\partial L}{\partial z^i}\right)+\left(g_{i\bar k}\frac{d^2\bar\gamma^k}{dt^2}+\frac{\partial^2 L}{\partial\bar z^k\partial\eta^i}\bar\eta^k\right)=0,\qquad i=\overline{1,n},\tag{5.4}$$

in which the brackets cannot be vanished like in [120] due to real homogeneity of L. The equation (5.4) of the geodesic curve $\gamma(t)$ can be written in the local coordinate as follows

$$g_{ij}\left(\frac{d^2\gamma^j}{dt^2}+2G^j(\gamma,\dot\gamma)\right)+\tilde T_i(\gamma,\dot\gamma)+g_{i\bar k}\left(\frac{d^2\bar\gamma^k}{dt^2}+2G^{\bar k}(\gamma,\dot\gamma)\right)-\theta^*_i(\gamma,\dot\gamma)=0,\ i=\overline{1,n},\tag{5.5}$$

where $\tilde T_i=\frac{1}{2}(2\delta_l g_{ij}-\delta_i g_{lj})\eta^l\eta^j-\frac{1}{2}g_{l\bar k}(\dot\partial_{\bar r}N^l_i)\bar\eta^k\bar\eta^r$ and $\theta^*_i=g_{l\bar k}T^l_{ji}\eta^j\bar\eta^k$.

The *holomorphic curvature* of a complex Finsler space is the analogue of the holomorphic sectional curvature from Hermitian geometry [1]. Following [1, 120], the holomorphic curvature of $\mathbb{R}$-complex Hermitian Finsler function F in direction η is

$$\mathcal{K}_F(z,\eta)=\frac{2}{\tilde L^2}g_{i\bar m}R^i_{j\bar h k}\eta^j\bar\eta^h\eta^k\bar\eta^m,\tag{5.6}$$

where $R^i_{j\bar h k}=-\delta_{\bar h}L^i_{jk}-(\delta_{\bar h}N^l_k)C^i_{jl}$ are the horizontal curvature coefficients of the Chern-Finsler connection D and $\tilde L=g_{i\bar m}\eta^k\bar\eta^m$.

$\mathbb{R}$-complex Randers metrics. Next, we have introduced an important class of $\mathbb{R}$-complex Hermitian Finsler metrics. Considering $z\in M$, $\eta\in T'_zM$, $\eta=\eta^i\frac{\partial}{\partial z^i}$, an $\mathbb{R}$-*complex Randers function* is defined by

$$F(z,\eta)=\alpha(z,\eta,\bar z,\bar\eta)+\beta(z,\eta,\bar z,\bar\eta),\tag{5.7}$$

where $\alpha^2(z,\eta,\bar z,\bar\eta)=\mathrm{Re}\{a_{ij}\eta^i\eta^j\}+a_{i\bar j}\eta^i\bar\eta^j$, $\beta(z,\eta,\bar z,\bar\eta)=\mathrm{Re}\{b_i\eta^i\}$, with $b=b_i(z)dz^i$ a differential $(1,0)$-form, $b\neq 0$, and $a_{i\bar j}=a_{i\bar j}(z)$ which comes from a Hermitian metric $a=a_{i\bar j}dz^i\otimes d\bar z^j$ on M [138, 90]. Note that α gives us an example of $\mathbb{R}$-complex Hermitian Finsler metric which is pure Hermitian and, by (5.1), $a_{i\bar j}(z)=\dot\partial_i\dot\partial_{\bar j}\alpha^2$ and $a_{ij}(z)=\dot\partial_i\dot\partial_j\alpha^2$.

In [41] we studied a particular $\mathbb{R}$-complex Hermitian Finsler space with Randers metrics, setting (5.7) with $a_{ij}=0$. Here we treat a_{ij} a bit more general, namely

$$a_{ij}=xb_ib_j,\tag{5.8}$$

where $x\in[0,1)$ is a real parameter. Considering the notations $b^k=a^{\bar jk}b_{\bar j}$ and $||b||^2=a^{\bar ji}b_ib_{\bar j}$ ($b_{\bar j}$ means $\bar b_j$), we prove (as in [43]) that the condition $||b||^2\in(0,\frac{1}{1-x})$ guarantees the positivity of F given by (5.7) with (5.8). The below shown proof is for $x=\frac{1}{2}$, i.e. $a_{ij}=\frac{1}{2}b_ib_j$.

Lemma 5.2.2. *Let $F=\alpha+\beta$ be an $\mathbb{R}$-complex Randers function with $a_{ij}=\frac{1}{2}b_ib_j$. Then, F is positive on $\widetilde{M}$ if and only if $||b||^2<2$. Moreover, any of these assertions implies $\alpha^2-\beta^2>0$.*

Proof. We assume that $F > 0$ on $\widetilde{M}$. This leads to $\sqrt{\text{Re}\{a_{ij}\eta^i\eta^j\} + a_{i\bar{j}}\eta^i\bar{\eta}^j} > -\text{Re}\{b_i\eta^i\}$ for all $\eta \neq 0$, which can be equivalently rewritten as $\sqrt{\frac{1}{2}\text{Re}\{(b_i\eta^i)^2\} + a_{i\bar{j}}\eta^i\bar{\eta}^j} > -\text{Re}\{b_i\eta^i\}$. Since $b \neq 0$, substituting in the last relation η^i with $-b^i$, it results that $||b||^2 < 2$. Conversely, under the assumptions $||b||^2 < 2$ and $\eta \neq 0$, the Cauchy-Schwarz inequality yields

$$|b_i\eta^i|^2 < a_{i\bar{j}}\eta^i\bar{\eta}^j||b||^2 < 2a_{i\bar{j}}\eta^i\bar{\eta}^j, \tag{5.9}$$

where $|b_i\eta^i|^2 = (b_i\eta^i)(b_{\bar{j}}\bar{\eta}^j)$.

The inequality (5.9) is equivalent to $\left(\text{Re}\{b_i\eta^i\}\right)^2 < \frac{1}{2}\text{Re}\{(b_i\eta^i)^2\} + a_{i\bar{j}}\eta^i\bar{\eta}^j$ which implies $\alpha^2 - \beta^2 > 0$, and $-\text{Re}\{b_i\eta^i\} < \left|\text{Re}\{b_i\eta^i\}\right| < \sqrt{\frac{1}{2}\text{Re}\{(b_i\eta^i)^2\} + a_{i\bar{j}}\eta^i\bar{\eta}^j}$, that is, F is positive. □

By some technical computations and making use of [27, Propositin 2.2], we get the result:

Proposition 5.2.3. *Let $F = \alpha + \beta$ be an $\mathbb{R}$-complex Randers function with $a_{ij} = \frac{1}{2}b_ib_j$. Then*
i) $g_{i\bar{j}} = \frac{F}{\alpha}(a_{i\bar{j}} - \frac{1}{2\alpha^2}l_il_{\bar{j}} + \frac{\alpha}{2F^3}\eta_i\eta_{\bar{j}})$ *and* $\tilde{L} = \frac{F^2(2\alpha-\beta)(\alpha^2-\beta^2+\varepsilon\bar{\varepsilon})}{2\alpha^3}$;
ii) $g_{ij} = \frac{F}{2\alpha}(b_ib_j - \frac{1}{\alpha^2}l_il_j + \frac{\alpha}{F^3}\eta_i\eta_j)$;
iii) $g^{\bar{j}i} = \frac{\alpha}{F}(a^{\bar{j}i} + \frac{4}{H}\zeta^i\bar{\zeta}^j - \frac{2\alpha}{FHJ}\vartheta^i\bar{\vartheta}^j)$;
iv) $\det(g_{i\bar{j}}) = \left(\frac{F}{\alpha}\right)^n\frac{J}{8\alpha^2}\det(a_{i\bar{j}})$, *where*

$$\begin{aligned}
\alpha^2 &= \tilde{l}_i\eta^i + \frac{\varepsilon^2 + \bar{\varepsilon}^2}{4}, \quad \tilde{l}_i = a_{i\bar{j}}\bar{\eta}^j, \quad \varepsilon = b_j\eta^j, \quad \varepsilon + \bar{\varepsilon} = 2\beta, \\
l_i &= \tilde{l}_i + \frac{\varepsilon}{2}b_i, \quad \eta_i = \frac{F}{\alpha}\left(\tilde{l}_i + \frac{\varepsilon + 2\alpha}{2}b_i\right), \\
\zeta^i &= \frac{\bar{\varepsilon}}{2}b^i + \eta^i, \quad \vartheta^i = \left[4F - \varepsilon(2 - ||b||^2)\right]\eta^i + 2F(F - \varepsilon)b^i, \\
H &= 4(\alpha^2 - \beta^2) + \varepsilon\bar{\varepsilon}(2 - ||b||^2), \quad J = H + 2\alpha F(2 + ||b||^2).
\end{aligned}$$

We notice that the condition $||b||^2 < 2$ also assures positive-definiteness of $g_{i\bar{j}}$. Thus, F given by (5.7) with $a_{ij} = \frac{1}{2}b_ib_j$ is an $\mathbb{R}$-complex Hermitian Randers metric (briefly $\mathbb{R}$-complex Randers metric).

5.3 Generalized Zermelo navigation under weak wind

Let (M, h) be an n-dimensional Hermitian manifold, where $h = h_{i\bar{j}}dz^i \otimes d\bar{z}^j$ is a pure Hermitian metric determined by the components $h_{j\bar{k}}(z) = h(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k})$ in the local coordinates $(z^k)_{k=\overline{1,n}}$ of $z \in M$ [138, 90]. The norm of the tangent vectors $\eta \in T'_zM$, $\eta = \eta^j\frac{\partial}{\partial z^j}$ with respect to h, (i.e. its h-length), is $||\eta||_h = \sqrt{h(\eta, \bar{\eta})} = \sqrt{h_{j\bar{k}}(z)\eta^j\bar{\eta}^k}$. We consider the Zermelo navigation problem on the imaginary sea represented by (M, h) in the presence of wind determined by a vector field $W \in T'_zM$, $W = W^j\frac{\partial}{\partial z^j}$. Like in the standard formulation of the problem [45], we denote by u the velocity of a ship in the absence of wind, but in order to follow the idea of the generalizations from [93, 96, 14, 15], it need not have h-unit length. Actually, as in [93] we admit that $||u||_h \in (||W||_h, 1]$. This implies the ship's relative speed $||u||_h$ may be space-dependent because also the wind speed $||W||_h$ has this property and it is more realistic model from a practical point of view.

Further on, by notation $(h, f(z), W)$ we mean *the generalized navigation data*, where $f(z) = ||u(z)||_h$. Function $f : M \to (||W||_h, 1]$ is assumed to be a smooth, positive and real valued which depends on z (also on $\bar{z}$), and W is a *weak wind*, i.e. $0 < ||W||_h < ||u(z)||_h \leq 1$. Due to the existence of perturbing vector field W a ship's resulting velocity will be represented by the tangent vector $v = u + W$ which starts from z. Substituting $u = v - W$ into $||u||_h = \sqrt{h(u, \bar{u})}$, it results that $||u||_h = \sqrt{||v||_h^2 - 2\text{Re}h(v, \bar{W}) + ||W||_h^2}$. Since $\text{Re}h(v, \bar{W}) = ||v||_h ||W||_h \cos\theta$, where here θ is the angle between v and W, the last relation can be rewritten as

$$||v||_h^2 - 2||v||_h ||W||_h \cos\theta - \psi = 0, \tag{5.10}$$

with $\psi = ||u||_h^2 - ||W||_h^2$.

In particular, on the calm Hermitian sea (M, h), i.e. $W = 0$, the solutions of the standard Zermelo navigation problem (with $||u||_h = 1$) are the geodesics of h. However, in the presence of wind ($W \neq 0$), the pure Hermitian metric h is deformed into a function F on $T'M$ such that $F(z, v) = 1$ [45]. Since $||W||_h < ||u||_h$, the quadratic equation (5.10) admits a positive root which is expressed by $||v||_h^2 = q + p$, where $q = \sqrt{[\text{Re}h(v, \bar{W})]^2 + ||v||_h^2 \psi}$ and $p = \text{Re}h(v, \bar{W})$. All these lead to

$$F(z, v) = ||v||_h^2 \frac{q - p}{q^2 - p^2} = \frac{\sqrt{[\text{Re}h(v, \bar{W})]^2 + \psi ||v||_h^2} - \text{Re}h(v, \bar{W})}{\psi}. \tag{5.11}$$

In order to obtain $F(z, \eta)$ for arbitrary non-zero vector $\eta \in T'_z M$, we take into consideration the fact that every non-zero η is written as a complex (in particular, real) multiple λ of some v, $\eta = \lambda v$. In general, the function F obtained in (5.11) is not complex homogeneous. In [14] a strong condition is required for the purpose of complex homogeneity and then solves the generalized problem. But without any additional condition, F is only real homogeneous, i.e. $F(z, \eta) = F(z, \lambda v) = \lambda F(z, v) = \lambda$, where $\lambda > 0$. And thus from (5.11) it is derived as the smooth function on $\widetilde{M}$

$$F(z, \eta) = \frac{\sqrt{[\text{Re}h(\eta, \bar{W})]^2 + ||\eta||_h^2 \psi}}{\psi} - \frac{\text{Re}h(\eta, \bar{W})}{\psi} \tag{5.12}$$

which may be an $\mathbb{R}$-complex Finsler metric. Actually, the resulting function F is a sum $F = \alpha + \beta$, with

$$\alpha = \sqrt{\frac{[\text{Re}h(\eta, \bar{W})]^2 + ||\eta||_h^2 \psi}{\psi^2}} = \sqrt{\text{Re}\{a_{ij}\eta^i \eta^j\} + a_{i\bar{j}}\eta^i \bar{\eta}^j}, \quad \beta = -\frac{\text{Re}h(\eta, \bar{W})}{\psi} = \text{Re}\{b_i \eta^i\},$$

where α is a pure Hermitian metric and $\text{Re}h(\eta, \bar{W}) \neq 0$ and

$$a_{i\bar{j}} = \frac{h_{i\bar{j}}}{\psi} + \frac{W_i W_{\bar{j}}}{2\psi^2}, \quad a_{ij} = \frac{W_i W_j}{2\psi^2}, \quad b_i = -\frac{W_i}{\psi}, \tag{5.13}$$

$W_i = h_{i\bar{j}} W^{\bar{j}}$, $W^{\bar{j}} = \bar{W}^j$, $W_{\bar{j}} = \bar{W}_j$. Some computations lead to the inverse of $a_{i\bar{j}}(z)$ from (5.13) and also other terms of F, i.e.

$$a^{\bar{j}i} = \psi(h^{\bar{j}i} - \frac{\psi}{\psi + f^2} W^i W^{\bar{j}}), \; b^i = \frac{-2\psi}{\psi + f^2} W^i, \; ||b||^2 = \frac{2||W||_h^2}{2f^2 - ||W||_h^2}, \quad \psi = \frac{f^2(2 - ||b||^2)}{2 + ||b||^2}. \tag{5.14}$$

Moreover, $||b||^2 \in (0,2)$ because the wind W is weak, i.e. $0 < ||W||_h < f(z) \leq 1$ and $\mathrm{Re}h(\eta,\bar{W}) \neq 0$. Hence, it ensures the positivity of the function $F(z,\eta)$ from (5.12) and the positive definitness of $g_{i\bar{j}} = \dot{\partial}_i\dot{\partial}_{\bar{j}}F^2$. Therefore, it is an $\mathbb{R}$-complex Randers metric with $x = \frac{1}{2}$.

Note that in absence of wind, formula (5.12) simplifies to $F(z,\eta) = \frac{1}{f(z)}||\eta||_h$. Thus, it is a pure Hermitian metric conformal to the background metric h, i.e. $g_{i\bar{j}} = \frac{1}{f^2(z)}h_{i\bar{j}}$. In particular, for $f(z) = 1$, $F(z,\eta) = ||\eta||_h$. Here we can observe the influence of the variable factor $f(z)$. More precisely, if $W = 0$, then the geodesics of the background Hermitian metric h are not necessarily the solutions to the problem as they are in the standard case, i.e. with $f(z) = 1$ [14].

The obtained solutions of the generalized ZNP can be summarized by the following proposition. Thus, we have

Proposition 5.3.1. *Let (M,h) be a Hermitian manifold. The generalized navigation data $(h, f(z), W)$ induce on the holomorphic tangent bundle $T'M$ the following:*
i) If $||W||_h \in (0,1)$, then the solution of the generalized ZNP is the $\mathbb{R}$-complex Randers metric $F = \alpha + \beta$ from (5.12);
ii) If $W = 0$, then the solution of the generalized ZNP is the pure Hermitian metric $F(z,\eta) = \frac{1}{f(z)}||\eta||_h$, conformal to h.

Since the generalized ZNP also produces pure Hermitian metrics which are conformal to the background metric h, it is natural to ask whether the converse is true. Namely, we have the following result

Corollary 5.3.2. *If $F = \frac{1}{\sqrt{\psi}}||\eta||_h$ is a conformal solution of the generalized ZNP, then $W = 0$ and $\sqrt{\psi} = f$.*

Proof. If $F(z,\eta) = \frac{1}{f(z)}||\eta||_h$ stands for a solution of the generalized ZNP, then by equation (5.12) we have $\mathrm{Re}h(\eta,\bar{W}) = 0$, for any η. This implies $W = 0$ and $\sqrt{\psi} = f$. □

Concluding, the generalized navigation data $(h, f(z), W)$ generate $\mathbb{R}$-complex Randers metrics and the corresponding geodesics are the solutions of the generalized ZNP.

The inverse problem is also of our interest. Namely, can every $\mathbb{R}$-complex Randers metric $F(z,\eta) = \alpha + \beta$ be achieved via the perturbation of the pure Hermitian metric h by some vector field W which satisfy $0 < ||W||_h < f(z) \leq 1$, where $f(z) = ||u||_h$? Although the proof for the inverse problem runs along similar lines likewise [14], there are subtle adjustments necessary to fit the argument to each new situation. Considering the $\mathbb{R}$-complex Randers metric $F(z,\eta) = \alpha+\beta$, with $\alpha = \sqrt{\mathrm{Re}\{a_{ij}\eta^i\eta^j\} + a_{i\bar{j}}\eta^i\bar{\eta}^j}$, $\beta = \mathrm{Re}\{b_i\eta^i\}$, $a_{ij} = \frac{1}{2}b_ib_j$, $b^i = a^{\bar{j}i}b_{\bar{j}}$ and $||b||^2 = b^ib_i \in (0,2)$, we construct h, $f(z)$ and W in the following way

$$h_{i\bar{j}}(z) = \tilde{\omega}(a_{i\bar{j}} - \frac{1}{2}b_ib_{\bar{j}}), \qquad ||u||_h = f(z), \qquad W^i(z) = \frac{-\left(\tilde{\omega} + f^2\right)b^i}{2\tilde{\omega}}, \tag{5.15}$$

where $\tilde{\omega} = \frac{f^2(2-||b||^2)}{2+||b||^2}$. By (5.15), some straightforward computations give

$$W_i = h_{i\bar{j}}W^{\bar{j}} = \tilde{\omega}(a_{i\bar{j}} - \frac{1}{2}b_ib_{\bar{j}})\frac{-\left(\tilde{\omega} + f^2\right)\bar{b}^j}{2\tilde{\omega}} = -\frac{f^2(2-||b||^2)}{2+||b||^2}b_i = -\tilde{\omega}b_i,$$

$$||W||_h^2 = h_{i\bar{j}}W^iW^{\bar{j}} = -\tilde{\omega}b_i\frac{-\left(\tilde{\omega} + f^2\right)b^i}{2\tilde{\omega}} = \frac{\left(\tilde{\omega} + f^2\right)||b||^2}{2} = \frac{2f^2||b||^2}{2+||b||^2}.$$

Since $||b||^2 < 2$, then $||W||_h < f(z)$. Moreover,

$$\psi = f^2 - \frac{2f^2||b||^2}{2+||b||^2} = \tilde{\omega}, \quad ||\eta||_h^2 = \tilde{\omega}(a_{i\bar{j}}\eta^i\bar{\eta}^j - \frac{\varepsilon\bar{\varepsilon}}{2}) = \tilde{\omega}(\alpha^2 - \beta^2), \quad \varepsilon = b_i\eta^i$$

and thus, $h(\eta, \bar{W}) = h_{i\bar{j}}\eta^i W^{\bar{j}} = W_i\eta^i = -\tilde{\omega}\varepsilon$ which turns out that $\mathrm{Re}h(\eta, \bar{W}) = -\tilde{\omega}\beta$. Replacing the obtained expressions for ψ, $||\eta||_h^2$ and $\mathrm{Re}h(\eta, \bar{W})$ with (5.12), we rediscover the $\mathbb{R}$-complex Randers metric $F(z, \eta) = \alpha + \beta$ under consideration.

Summarizing, we get

Theorem 5.3.3. *An $\mathbb{R}$-complex Hermitian Finsler metric F is of Randers type, i.e. $F = \alpha+\beta$ with* (5.13)*, if and only if it solves the generalized Zermelo navigation problem on a Hermitian manifold (M, h), with space-dependent ship's relative speed $||u(z)||_h \leq 1$ and under action of weak wind W. Moreover, F is a pure Hermitian metric conformal to h, with the conformal factor $\frac{1}{||u(z)||_h}$, if and only if $W = 0$.*

Further on, an $\mathbb{R}$-complex Randers metric $F = \alpha + \beta$ with (5.13) which describe the Zermelo deformation of the Hermitian metric h by weak wind W, with a space-dependent ship's relative speed $||u||_h$ will be called briefly *W-Zermelo deformation.*

5.4 W-Zermelo metric

Next, our aim is to investigate how some properties of a Hermitian metric h, i.e. Kähler property and the holomorphic sectional curvature, behave by the Zermelo deformation, if the weak wind W is a special vector field. For that attempt, first of all we need to find the connections among the geometric tools corresponding to the metric h and W-Zermelo deformation F.

5.4.1 From h to W-Zermelo metric F via α

The study of $\mathbb{R}$-complex Hermitian Randers metrics was developed in a few papers [41, 22], but only in particular case $a_{ij} = 0$. In contrast, for the investigation on the Zermelo deformation we need to have developed geometry of $\mathbb{R}$-complex Randers metrics with $a_{ij} = \frac{1}{2}b_ib_j$. Some of the elements and properties of such metrics were pointed out in Section 5.2. Now, we come with some new results from point of view of our purposes. Once obtained the metric tensor of an $\mathbb{R}$-complex Randers metric F (Proposition 5.2.3) it is a technical computation to give the local coefficients of the Chern-Finsler complex nonlinear connection, $N_j^i = g^{\bar{m}i}\frac{\partial^2 F^2}{\partial z^k\partial\bar{\eta}^m}$. Certainly, it involves some simple calculus which leads to

$$N_j^i = \overset{a}{N_j^i} + \frac{2}{J}(\overset{a}{\delta_j}\beta)\varkappa^i + \frac{1}{J}\frac{\partial b_{\bar{r}}}{\partial z^j}\mathbf{k}^{\bar{r}i}, \tag{5.16}$$

where J is defined in Proposition 5.2.3 and

$$\begin{aligned}
&\overset{a}{N_j^i} = a^{\bar{m}i}\frac{\partial^2\alpha^2}{\partial z^j\partial\bar{\eta}^m} = a^{\bar{m}i}\left(\frac{\partial a_{s\bar{m}}}{\partial z^j}\eta^s + \frac{\bar{\varepsilon}}{2}\frac{\partial b_{\bar{m}}}{\partial z^j}\right) + \frac{1}{2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r b^i,\\
&2(\overset{a}{\delta_j}\beta) = \frac{\partial\beta}{\partial z^j} - \overset{a}{N_j^k}(\dot{\partial}_k\beta) = \frac{\partial\bar{b}^r}{\partial z^j}\tilde{l}_{\bar{r}} + \frac{1}{2}\frac{\partial b_{\bar{r}}}{\partial z^j}\left[(2-||b||^2)\bar{\eta}^r - \bar{\varepsilon}b^{\bar{r}}\right],\\
&\mathbf{k}^{\bar{r}i} = \alpha J a^{\bar{r}i} + 2(2\beta + \alpha||b||^2)\eta^i\bar{\eta}^r - \alpha\varkappa^i b^{\bar{k}} - 2F(2\alpha - \bar{\varepsilon})b^i\bar{\eta}^r,\\
&\varkappa^i = 2(2\alpha - \varepsilon)\eta^i + (2F\alpha - \varepsilon\bar{\varepsilon})b^i.
\end{aligned} \tag{5.17}$$

Here the differential $(1,0)$-form $b = b_i(z)dz^i$ is said to be *biholomorphic*, if the functions b_i and b^i are holomorphic, i.e. $\frac{\partial b_i}{\partial \bar{z}^k} = \frac{\partial b^i}{\partial \bar{z}^k} = 0$, for all $i, k = \overline{1,n}$.

Lemma 5.4.1. *The differential* $(1,0)$*-form* b *is biholomorphic if and only if* $\overset{a}{\delta}_j \beta = 0$.

Proof. Since $\overset{a}{\delta}_j = \frac{\partial}{\partial z^j} - \overset{a}{N}{}^i_j \frac{\partial}{\partial \eta^i}$, we get $2(\overset{a}{\delta}_j \beta) = \frac{\partial \bar{b}^r}{\partial z^j} \tilde{l}_{\bar{r}} + \frac{1}{2} \frac{\partial b_{\bar{r}}}{\partial z^j} \left[(2 - ||b||^2)\bar{\eta}^r - \bar{\varepsilon} b^{\bar{r}}\right]$, and thus the direct implication results immediately. Conversely, the condition $\overset{a}{\delta}_j \beta = 0$ can be rewritten as

$$\frac{\partial \bar{b}^r}{\partial z^j} \tilde{l}_{\bar{r}} + \frac{1}{2} \frac{\partial b_{\bar{r}}}{\partial z^j} \left[(2 - ||b||^2)\bar{\eta}^r - \bar{\varepsilon} b^{\bar{r}}\right] = 0. \tag{5.18}$$

Differentiating (5.18) with respect to η results that $\frac{\partial b^{\bar{m}}}{\partial z^j} a_{i\bar{m}} = 0$. Thus, $\frac{\partial b^{\bar{m}}}{\partial z^j} = 0$, and so b^i are holomorphic. Moreover, (5.18) becomes $\frac{\partial b_{\bar{r}}}{\partial z^j} \left[(2 - ||b||^2)\bar{\eta}^r - \bar{\varepsilon} b^{\bar{r}}\right] = 0$. Now, differentiating the last relation with respect to $\bar{\eta}$ yields

$$\frac{\partial b_{\bar{m}}}{\partial z^j}(2 - ||b||^2) - b^{\bar{r}} \frac{\partial b_{\bar{r}}}{\partial z^j} b_{\bar{m}} = 0 \tag{5.19}$$

which contracted by $b^{\bar{m}}$ leads to $b^{\bar{m}} \frac{\partial b_{\bar{m}}}{\partial z^j}(1 - ||b||^2) = 0$. We distinguish two cases. First, if $||b||^2 \neq 1$, then $b^{\bar{m}} \frac{\partial b_{\bar{m}}}{\partial z^j} = 0$. By (5.19) it follows that $\frac{\partial b_{\bar{m}}}{\partial z^j} = 0$, i.e. b_i are holomorphic. Second, if $||b||^2 = 1$, then $b^{\bar{m}} \frac{\partial b_{\bar{m}}}{\partial z^j} = 0$, because b^i are holomorphic. Using again by (5.19), it turns out that b_i are holomorphic. □

Lemma 5.4.2. *Let* $F = \alpha + \beta$ *be an* $\mathbb{R}$*-complex Randers metric with* $a_{ij} = \frac{1}{2} b_i b_j$ *and* b *biholomorphic. Then,*
i) $N^i_j = \overset{a}{N}{}^i_j$ *and* $G^i = \overset{a}{G}{}^i$, *where* $\overset{a}{G}{}^i = \frac{1}{2} \overset{a}{N}{}^i_j \eta^j$ *are the spray coefficients of* α*;*
ii) $\frac{\partial \overset{a}{N}{}^i_j}{\partial \bar{z}^k} b_i = 0$*;*
iii) α *is Kähler if and only if* F *is strongly Kähler;*
iv) the holomorphic curvature in direction η *corresponding to* F *is*

$$\mathcal{K}_F(z,\eta) = P(\alpha^2 - \beta^2 + \frac{\varepsilon\bar{\varepsilon}}{2})^2 \mathcal{K}_\alpha(z,\eta), \tag{5.20}$$

where $P = \frac{\alpha[2\alpha^2(2\alpha-\beta)F - 3(\alpha-\beta)(\alpha^2-\beta^2+\varepsilon\bar{\varepsilon})\beta]}{F^3(2\alpha-\beta)^2(\alpha^2-\beta^2+\varepsilon\bar{\varepsilon})^2}$ *and* $\mathcal{K}_\alpha(z,\eta)$ *is the holomorphic curvature in direction* η *which corresponds to* α.

Proof. Since b is biholomorphic, the formula (5.16) is reduced to $N^i_j = \overset{a}{N}{}^i_j = a^{\bar{m}i} \frac{\partial a_{s\bar{m}}}{\partial z^j} \eta^s$. The last relation leads to $\frac{\partial \overset{a}{N}{}^i_j}{\partial \bar{z}^k} b_i = \frac{\partial}{\partial \bar{z}^k}(\overset{a}{N}{}^i_j b_i) = \frac{\partial}{\partial \bar{z}^k}(\frac{\partial b_s}{\partial z^j})\eta^s = \frac{\partial}{\partial z^j}(\frac{\partial b_s}{\partial \bar{z}^k})\eta^s = 0$, i.e. ii). Moreover, $N^i_j = \overset{a}{N}{}^i_j$ implies $T^i_{jk} = \overset{a}{T}{}^i_{jk}$ which justifies iii), where T^i_{jk} was defined in Section 5.2 and $\overset{a}{T}{}^i_{jk} = \dot{\partial}_j \overset{a}{N}{}^i_k - \dot{\partial}_k \overset{a}{N}{}^i_j$. Now, using i), ii), iii), (5.6) and Proposition 5.2.3, by straightforward computations we obtain (5.20). □

Next step is to find some links between a and h in terms of the generalized navigation data $(h, f(z), W)$. Remark that the pure Hermitian metric a is only an intermediary step on the way to W-Zermelo deformation F. Thus, starting with the relations between the Hermitian metrics a and h, i.e. (5.13) and (5.14), after a straightforward computation we are led to

$$\overset{a}{N}{}^i_j = \overset{h}{N}{}^i_j - \frac{\partial \log \psi}{\partial z^j}\eta^i + \frac{W_0 + W_{\bar{0}}}{2\psi} h^{\bar{m}i}\left(\frac{\partial W_{\bar{m}}}{\partial z^j} - \frac{\partial \log \psi}{\partial z^j} W_{\bar{m}}\right) + \frac{1}{2} W^i \overset{a}{\delta}_j \frac{W_0 + W_{\bar{0}}}{\psi}, \tag{5.21}$$

where $\overset{h}{N}{}^i_j = h^{\bar{m}i}\frac{\partial h_{r\bar{m}}}{\partial z^j}\eta^r$ denotes the coefficients of Chern-Finsler complex nonlinear connections corresponding to the Hermitian metric h and the indices 0 and $\bar{0}$ mean the contractions by η and $\bar{\eta}$, respectively.

5.4.2 Holomorphic Zermelo deformation

The vector field W (or the weak wind in our approach) is the section of $T'M$ and in terms of local complex coordinates $(z^k)_{k=\overline{1,n}}$ it is $W = W^k \frac{\partial}{\partial z^k}$. Following [110], we say that the weak wind W is *the holomorphic vector field*, if the components W^k are the holomorphic functions, i.e. $\frac{\partial W^k}{\partial \bar{z}^r} = 0$, for all $k = \overline{1,n}$. Further on, we pay our attention to the holomorphic vector fields W with

$$W^k_{|j} = \frac{\partial \log f^2}{\partial z^j} W^k, \tag{5.22}$$

where $W^k_{|j} = \frac{\partial W^k}{\partial z^j} + h^{\bar{m}k}\frac{\partial h_{r\bar{m}}}{\partial z^j} W^r$ and they will be called *f-holomorphic*. In particular, if h is Kähler, then $h^{\bar{m}k}\frac{\partial h_{r\bar{m}}}{\partial z^j} = \Gamma^k_{jr}$ which denotes the Christoffel symbols corresponding to h.

Lemma 5.4.3. *Let $(h, f(z), W)$ be the generalized navigation data. If W is f-holomorphic, then*
i) $\frac{\partial W_{\bar{m}}}{\partial z^j} = \frac{\partial \log f^2}{\partial z^j} W_{\bar{m}}$;
ii) $\frac{\partial \log f^2}{\partial z^j} = \frac{\partial \log \psi}{\partial z^j} = \frac{\partial \log ||W||^2_h}{\partial z^j}$;
iii) $||W||^2_h = cf^2$, $\psi = (1-c)f^2$, *where c is a constant,* $c \in (0,1)$.

Proof. We assume that W is an f-holomorphic vector field. Differentiating $W_{\bar{m}} = h_{k\bar{m}}W^k$ and $||W||^2_h = W_{\bar{m}}\bar{W}^m$ with respect to z yield $\frac{\partial W_{\bar{m}}}{\partial z^j} = \frac{\partial \log f^2}{\partial z^j} W_{\bar{m}}$, i.e. i), and moreover, it follows that $\frac{\partial \log f^2}{\partial z^j} = \frac{\partial \log ||W||^2_h}{\partial z^j}$. Since $\psi = f^2 - ||W||^2_h$, its differentiation with respect to z gives

$$\frac{\partial \log \psi}{\partial z^j} = \frac{\partial f^2}{\partial z^j} - \frac{\partial \log ||W||^2_h}{\partial z^j}$$

which leads to $\frac{\partial \log f^2}{\partial z^j} = \frac{\partial \log \psi}{\partial z^j}$, and thus ii). Having integrated ii) with respect to z yields the relations in iii), where c is the constant of integration. Since W is weak, it results that $c \in (0,1)$. □

Corollary 5.4.4. *Let $(h, f(z), W)$ be the generalized navigation data. Then the vector field W is f-holomorphic if and only if the differential $(1,0)$-form b is biholomorphic, with $b_i = -\frac{W_i}{\psi}$.*

Proof. Whether or not W is f-holomorphic and b is biholomorphic, since $b^{\bar{m}} = \frac{-2\psi}{\psi + f^2} W^{\bar{m}}$,

$b_{\bar{m}} = -\frac{W_{\bar{m}}}{\psi}$ and $||b||^2 = b^{\bar{m}} b_{\bar{m}} = \frac{2||W||_h^2}{2f^2 - ||W||_h^2}$, it follows that

$$\begin{aligned} \frac{\partial b^{\bar{m}}}{\partial z^j} &= \frac{2\psi f^2}{(\psi + f^2)^2}\left(\frac{\partial \log f^2}{\partial z^j} - \frac{\partial \log \psi}{\partial z^j}\right) W^{\bar{m}} - \frac{2\psi}{\psi + f^2}\frac{\partial W^{\bar{m}}}{\partial z^j}, \\ \frac{\partial b_{\bar{m}}}{\partial z^j} &= \frac{1}{\psi}\left(\frac{\partial \log \psi}{\partial z^j} W_{\bar{m}} - \frac{\partial W_{\bar{m}}}{\partial z^j}\right), \\ \frac{\partial b^{\bar{m}}}{\partial z^j} b_{\bar{m}} + b^{\bar{m}} \frac{\partial b_{\bar{m}}}{\partial z^j} &= \frac{4f^2||W||_h^2}{(2f^2 - ||W||_h^2)^2}\left(\frac{\partial \log f^2}{\partial z^j} - \frac{\partial \log ||W||_h^2}{\partial z^j}\right). \end{aligned} \tag{5.23}$$

If W is f-holomorphic, then from (5.23) and Lemma 5.4.3 it results that b is biholomorphic. Conversely, since b is biholomorphic, the conditions (5.23) lead to the fact that W is f-holomorphic. □

Corollary 5.4.5. *Let $(h, f(z), W)$ be the generalized navigation data. If the vector field W is f-holomorphic, then the local coefficients of the Chern-Finsler complex nonlinear connection corresponding to W-Zermelo deformation F are given by*

$$N_j^i = \overset{h}{N_j^i} - \frac{\partial \log f^2}{\partial z^j}\eta^i. \tag{5.24}$$

In particular, if f is constant, then $N_j^i = \overset{h}{N_j^i}$, $\frac{\partial W_{\bar{m}}}{\partial z^j} = 0$ and $W_{|j}^k = 0$.

Proof. The proof follows from Corollary 5.4.4, Lemma 5.4.3 and (5.21). □

Further on, an f-holomorphic weak wind W, with $f = const.$, is said to be *biholomorphic*.

Theorem 5.4.6. *Let (M, h) be an n-dimensional Kähler manifold, $n \geq 2$, and $(h, f(z), W)$ be the generalized navigation data, with W an f-holomorphic vector field. Then, W is biholomorphic if and only if W-Zermelo deformation F is strongly Kähler.*

Proof. Under our assumptions the direct implication results by Corollary 5.4.5. Conversely, since F is strongly Kähler, the equation (5.24) implies that $\frac{\partial \log f^2}{\partial z^j}\delta_k^i - \frac{\partial \log f^2}{\partial z^k}\delta_j^i = 0$. Summing with $i = j$, we deduce that $(n-1)\frac{\partial \log f^2}{\partial z^k} = 0$. Thus, f is constant. □

Next, also by the assumption that the weak wind W is f-holomorphic and using (5.24), some computations lead to the link between holomorphic curvatures in direction η, corresponding to h and a. Namely, we obtain

$$\mathcal{K}_\alpha(z, \eta) = \frac{\tilde{h}^4}{(1-c)f^2(\alpha^2 - \beta^2 + \frac{\varepsilon\bar{\varepsilon}}{2})^2}\left(\mathcal{K}_h(z, \eta) + \frac{2}{\tilde{h}^2}\frac{\partial^2 \log f^2}{\partial z^j \partial \bar{z}^m}\eta^j \bar{\eta}^m\right), \tag{5.25}$$

where $\mathcal{K}_h(z, \eta) = \frac{2}{\tilde{h}^2} h_{i\bar{m}} \overset{h}{R^i}_{j\bar{r}k} \eta^j \bar{\eta}^r \eta^k \bar{\eta}^m$, $\overset{h}{R^i}_{j\bar{r}k} = -\frac{\partial}{\partial \bar{z}^r}(\dot{\partial}_j \overset{h}{N_k^i})$ and $\tilde{h} = ||\eta||_h = \sqrt{\psi(\alpha^2 - \beta^2)}$. We thus have the following result.

Theorem 5.4.7. *Let (M, h) be an n-dimensional Hermitian manifold and $(h, f(z), W)$ be the generalized navigation data, with W an f-holomorphic vector field. Then, the holomorphic curvature in direction η, corresponding to W-Zermelo deformation F is*

$$\mathcal{K}_F(z, \eta) = \frac{\tilde{h}^4 P}{(1-c)f^2}\left(\mathcal{K}_h(z, \eta) + \frac{2}{\tilde{h}^2}\frac{\partial^2 \log f^2}{\partial z^j \partial \bar{z}^m}\eta^j \bar{\eta}^m\right), \tag{5.26}$$

where $c \in (0, 1)$. If $\frac{\partial \log f^2}{\partial z^j}$ is a holomorphic function, then $\mathcal{K}_F(z, \eta) = \frac{\tilde{h}^4 P}{(1-c)f^2}\mathcal{K}_h(z, \eta)$.

Proof. Substituting (5.25) into (5.20), we obtain (5.26). The particular case is obvious. □

Theorem 5.4.8. *Let (M,h) be an n-dimensional Hermitian manifold and $(h,f(z),W)$ be the generalized navigation data, with W an f-holomorphic vector field. If the holomorphic curvature $\mathcal{K}_F(z,\eta)$ of W-Zermelo deformation F depends only on z, $\mathcal{K}_F(z,\eta)=k(z)$, then $k(z)=0$ and*

$$\mathcal{K}_h(z,\eta) = -\frac{2}{\tilde{h}^2}\frac{\partial^2 \log f^2}{\partial z^j \partial \bar{z}^m}\eta^j\bar{\eta}^m. \tag{5.27}$$

Proof. Since $A=\frac{\tilde{h}^4}{(1-c)f^2}\left(\mathcal{K}_h(z,\eta)+\frac{2}{\tilde{h}^2}\frac{\partial^2\log f^2}{\partial z^j\partial\bar{z}^m}\eta^j\bar{\eta}^m\right)$ is a polynomial in η and $\bar{\eta}$ of second degree and $\mathcal{K}_F(z,\eta)=k(z)$, formula (5.26) can be rewritten as

$$\alpha[(4\alpha^4-\beta^4+\alpha^2\beta^2)k(z)-(4\alpha^4-3\beta^4+\alpha^2\beta^2+3\varepsilon\bar{\varepsilon}\beta^2)A]$$

$$+\beta[(8\alpha^4+\beta^4-5\alpha^2\beta^2)k(z)+(\alpha^4-3\alpha^2\beta^2+3\varepsilon\bar{\varepsilon}\alpha^2)A]=0,$$

which contains an irrational part and a rational part. This implies the following homogeneous linear system

$$\begin{cases}(4\alpha^4-\beta^4+\alpha^2\beta^2)k(z)-(4\alpha^4-3\beta^4+\alpha^2\beta^2+3\varepsilon\bar{\varepsilon}\beta^2)A=0\\(8\alpha^4+\beta^4-5\alpha^2\beta^2)k(z)+(\alpha^4-3\alpha^2\beta^2+3\varepsilon\bar{\varepsilon}\alpha^2)A=0\end{cases}, \tag{5.28}$$

with the unknowns $k(z)$ and A. Since the associated determinant is nonzero, system (5.28) admits only the null solution, i.e. $k(z)=0$ and $A=0$. □

Note that, if there exists an f-holomorphic vector field W such that (5.27) holds, then by (5.26) it follows that $\mathcal{K}_F(z,\eta)=0$. Owing to Theorem 5.4.7, we have the following corollaries

Corollary 5.4.9. *Let (M,h) be an n-dimensional Hermitian manifold and $(h,f(z),W)$ be the generalized navigation data, with an f-holomorphic vector field W, and $\frac{\partial^2\log f^2}{\partial z^j}$ is a holomorphic function. Then, $\mathcal{K}_h(z,\eta)=0$ if and only if $\mathcal{K}_F(z,\eta)=0$.*

Corollary 5.4.10. *Let (M,h) be an n-dimensional Kähler manifold and $(h,f(z),W)$ be the generalized navigation data, with a biholomorphic vector field W. Then, the geodesic curves $\gamma(t)$ corresponding to W-Zermelo deformation F are the solutions of the system*

$$g_{ij}\left(\frac{d^2\gamma^j}{dt^2}+2\overset{h}{G^j}(\gamma,\dot{\gamma})\right)+g_{i\bar{k}}\left(\frac{d^2\gamma^{\bar{k}}}{dt^2}+2\overset{h}{G^{\bar{k}}}(\gamma,\dot{\gamma})\right)+\tilde{T}_i(\gamma,\dot{\gamma})=0, \tag{5.29}$$

where $\tilde{T}_i(z,\eta)=\frac{1}{8\alpha^3}\frac{\partial a_{j\bar{m}}}{\partial z^l}\{(\tilde{p}b_i+q\tilde{l}_i)b^{\bar{m}}+(\tilde{q}b_i-4\beta)\tilde{l}_i]\bar{\eta}^m\}\eta^j\eta^l$, $i=\overline{1,n}$, $\tilde{q}=2(2\alpha^2-\beta\varepsilon)$ *and* $\tilde{p}=8\alpha^3+4\alpha^2\beta-\beta\varepsilon^2+4\alpha^2\varepsilon$.

Proof. Under our assumptions, making use of (5.5), it yields (5.29). □

Lastly, we exemplify the Zermelo deformation by a few relevant models based on the generalized navigation data $(h,f(z),W)$, where W is an f-holomorphic vector field.

Example 5.4.11 Let h be the standard Euclidean metric ($h_{i\bar{j}}=\delta_{i\bar{j}}$) on $\mathbb{C}^n$ and let W be a weak wind with constant components, i.e. $W^k=\lambda_k$, $\lambda_k\in\mathbb{C}$, $k=\overline{1,n}$, such that

$||W||_h^2 = \sum_{k=1} |\lambda_k|^2 < f^2(z) = 1$. These lead to $\psi = 1 - \sum_{k=1} |\lambda_k|^2 = const.$, $b_i = \frac{1}{\psi}\bar{\lambda}_i$, $a_{i\bar{j}} = \frac{\delta_{i\bar{j}}}{\psi} + \frac{1}{2\psi^2}\bar{\lambda}_i\lambda_k$ and thus, $g_{i\bar{j}}$ and g_{ij} corresponding to W-Zermelo deformation F depend only on η, i.e. F is locally Minkowski. Moreover, it follows that $\mathcal{K}_F = \mathcal{K}_h = 0$.

In what follows we consider the Hermitian manifold M represented by $\mathbb{C}^2$ or a subset of $\mathbb{C}^2$. As in dimension two we denote the local position coordinates (z^1, z^2) by (z, w), and the fibres (η^1, η^2) by (η, ϱ).

Example 5.4.12 On $\mathbb{C}^2$ we consider the generalized navigation data $(h,\ f = 1,\ W = \frac{e^{-w}}{2}\frac{\partial}{\partial w})$ with $\left(h_{i\bar{j}}(z,w)\right) = \begin{pmatrix} e^{2\text{Re}z} & 0 \\ 0 & e^{2\text{Re}w} \end{pmatrix}$. These imply that $||W||_h^2 = \frac{1}{4}$, $W_1 = 0$, $W_2 = \frac{e^{w}}{2}$, $\psi = \frac{3}{4}$ and then, $b_1 = 0$, $b_2 = \frac{-2e^{w}}{3}$ as well as $\left(a_{i\bar{j}}(z,w)\right) = \frac{2}{9}\begin{pmatrix} 6e^{2\text{Re}z} & 0 \\ 0 & 7e^{2\text{Re}w} \end{pmatrix}$.

Consequently, we obtain $b^1 = 0$, $b^2 = -\frac{9e^{-w}}{4}$ and $||b||^2 = \frac{3}{2}$. Thus, W is biholomorphic and W-Zermelo deformation $F = \alpha + \beta$ has the components

$$\alpha^2 = \frac{2}{9}\left[6e^{z+\bar{z}}\,|\eta|^2 + 7e^{w+\bar{w}}\,|\varrho|^2 + \text{Re}(e^{2w}\varrho^2)\right], \qquad \beta = -\frac{2}{3}\text{Re}(e^{w}\varrho).$$

Since $N_1^1 = \overset{h}{N_1^1} = \eta^1$, $N_2^1 = \overset{h}{N_2^1} = N_1^2 = \overset{h}{N_1^2} = 0$, $N_2^2 = \overset{h}{N_2^2} = \eta^2$, it follows that F is strongly Kähler and $\mathcal{K}_F = \mathcal{K}_h = 0$.

Example 5.4.13 We consider the Hartogs triangle $\Delta = \left\{(z,w) \in \mathbb{C}^2,\ |w| < |z| < 1\right\}$ with the generalized navigation data $\left(h_{i\bar{j}}(z,w)\right) = \begin{pmatrix} |z|^2 & 0 \\ 0 & 1 \end{pmatrix}$, $f^2 = |z|^2$ and $W = \frac{z}{2}\frac{\partial}{\partial w}$.

It turns out that $||W||_h^2 = \frac{|z|^2}{4}$, $W_1 = b_1 = 0$, $W_2 = \frac{\bar{z}}{2}$, $\psi = \frac{3|z|^2}{4}$, $b_2 = -\frac{2}{3z}$ and $\left(a_{i\bar{j}}(z,w)\right) = \frac{2}{9}\begin{pmatrix} 6 & 0 \\ 0 & \frac{7}{|z|^2} \end{pmatrix}$. In this case W is f-holomorphic and the components of W-Zermelo deformation $F = \alpha + \beta$ are

$$\alpha^2 = \frac{2}{9}\left(6\,|\eta|^2 + \frac{7\,|\varrho|^2}{|z|^2} + \text{Re}\frac{\varrho^2}{z^2}\right), \qquad \beta = -\frac{2}{3}\text{Re}\frac{\varrho}{z}. \tag{5.30}$$

Moreover, by Corollary 5.4.9, $\mathcal{K}_F = 0$ because $\mathcal{K}_h = 0$ and $\frac{\partial \log f^2}{\partial z} = \frac{1}{z}$.

Example 5.4.14 On the Hartogs triangle $\Delta = \left\{(z,w) \in \mathbb{C}^2,\ |w| < |z| < 1\right\}$ we consider the pure Hermitian metric $\left(h_{i\bar{j}}(z,w)\right) = \begin{pmatrix} |z|^2 - |w|^2 & 0 \\ 0 & 1 \end{pmatrix}$ and the same data as in the last example, $f^2 = |z|^2$ and $W = \frac{z}{2}\frac{\partial}{\partial w}$.

These lead to the same tools, namely, $||W||_h^2 = \frac{|z|^2}{4}$, $W_1 = b_1 = 0$, $W_2 = \frac{\bar{z}}{2}$, $\psi = \frac{3|z|^2}{4}$, $b_2 = -\frac{2}{3z}$, excepting $a_{i\bar{j}}$ which here is $\left(a_{i\bar{j}}(z,w)\right) = \frac{2}{9|z|^2}\begin{pmatrix} 6(|z|^2 - |w|^2) & 0 \\ 0 & 7 \end{pmatrix}$.

Thus, this also influences W-Zermelo deformation $F = \alpha + \beta$ which has the same β as in (5.30) and

$$\alpha^2 = \frac{2}{9}\left[\frac{6(|z|^2 - |w|^2)}{|z|^2}\,|\eta|^2 + \frac{7\,|\varrho|^2}{|z|^2} + \text{Re}\frac{\varrho^2}{z^2}\right].$$

Although for any $j,m=1,2$ we have $\frac{\partial^2 \log f^2}{\partial z^j \partial \bar{z}^m}=0$, the holomorphic curvature of F is not vanishing ($\mathcal{K}_F \neq 0$) because $\mathcal{K}_h=\frac{2|\eta|^2|z\varrho-w\eta|^2}{(|z|^2-|w|^2)[(|z|^2-|w|^2)|\eta|^2+|\varrho|^2]^2}$.

Part II. Extensions of Matsumoto's slope-of-a-mountain problem

Chapter 6

Rudiments of real Finsler geometry

In this chapter we briefly recall the notions and general facts from Riemann-Finsler geometry that are needed for presenting and proving our results. For more details, see, e.g. [71, 58, 45, 127, 155, 89, 154, 61, 85].

6.1 Finsler manifolds

Let (M, h) be a Riemannian manifold, where M is an n-dimensional C^{∞}-manifold, $n > 1$, and h is a Riemannian metric on M. Let T_xM be the tangent space at $x \in M$ and (x^i), $i = 1, ..., n$ be the local coordinate system on a local chart in $x \in M$. The set $\left\{\frac{\partial}{\partial x^i}\right\}$, $i = 1, ..., n$ denotes the natural basis for the tangent bundle $TM = \bigcup\limits_{x \in M} T_xM$ which is itself a C^{∞}-manifold. Thus, for every $y \in T_xM$, one has $y = y^i \frac{\partial}{\partial x^i}$ and the coordinates on a local chart in $(x, y) \in TM$ are denoted by (x^i, y^i), $i = 1, ..., n$.

A natural generalization of a Riemannian metric is a *Finsler metric*. Specifically, the pair (M, F) is a Finsler manifold if $F : TM \to [0, \infty)$ is a continuous function with the following properties:

i) F is a C^{∞}-function on the slit tangent bundle $TM_0 = TM \backslash \{0\}$;

ii) F is positively homogeneous of degree one with respect to y, i.e. $F(x, cy) = cF(x, y)$, for all $c > 0$;

iii) the Hessian $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite for all $(x, y) \in TM_0$.

Denoting by $I_F = \{(x, y) \in TM \mid F(x, y) = 1\}$ the indicatrix of F, one can remark that the property iii) refers to the fact that I_F is strongly convex. In particular, the Finsler metric F is a Riemannian metric if and only if $g_{ij}(x, y)$ does not depend on y, i.e. $g_{ij}(x, y) = g_{ij}(x)$.

Let $\mathcal{A}$ be a conic open subset of TM_0. According to [61, 87, 88], this means that for each $x \in M$, $\mathcal{A}_x = \mathcal{A} \cap T_xM$ is a conic subset, i.e. if $y \in \mathcal{A}_x$, then $\lambda y \in \mathcal{A}_x$ for every $\lambda > 0$. In particular, a *conic Finsler metric* is a Finsler metric on $\mathcal{A}$, i.e. $F : \mathcal{A} \to [0, \infty)$ is a continuous function satisfying i), ii) and iii) for all $(x, y) \in \mathcal{A}$ (see [61, 87]).

A smooth vector field on TM_0, locally expressed by $S = y^i \frac{\partial}{\partial x^i} - 2\mathcal{G}^i \frac{\partial}{\partial y^i}$, is called a *spray* on M. The functions $\mathcal{G}^i = \mathcal{G}^i(x, y)$, $i = 1, ..., n$ are positively homogeneous of degree two with respect to y, i.e. $\mathcal{G}^i(x, cy) = c^2\mathcal{G}^i(x, y)$, for all $c > 0$, and they are called the *spray coefficients* [71]. In the case where the spray is induced by a Finsler metric $F = \sqrt{g_{ij}(x, y)y^iy^j}$, the spray coefficients are given by

$$\mathcal{G}^i(x,y) = \frac{1}{4}g^{il}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\} = \frac{1}{4}g^{il}\left(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)y^j y^k, \tag{6.1}$$

(g^{il}) being the inverse matrix of (g_{il}).

Let us consider a regular piecewise C^∞-curve on M, $\gamma : [0,1] \to M$, $\gamma(t) = (\gamma^i(t))$, $i = 1, ..., n$, where the velocity vector of γ is denoted by $\dot{\gamma}(t) = \frac{d\gamma}{dt}$. The curve γ is called *F-geodesic* if $\dot{\gamma}(t)$ is parallel along the curve, i.e. in the local coordinates, $\gamma^i(t)$, $i = 1, ..., n$ are the solutions of the ODE system

$$\ddot{\gamma}^i(t) + 2\mathcal{G}^i(\gamma(t), \dot{\gamma}(t)) = 0. \tag{6.2}$$

It is worthwhile to mention that Zermelo's navigation, apart from the fact that it is a classic optimal control problem, provides a technique to construct a new Finsler metric by perturbing a given Finsler metric (the so called background metric) by a vector field W, i.e. a time-independent wind on a manifold M, under some constraints. In particular, by considering that the background metric is a Riemannian one, denoted by h, the Randers metric solves Zermelo's problem of navigation in the case of weak wind W, i.e. $||W||_h < 1$ [45, 71]. When W is a critical wind, i.e. $||W||_h = 1$, the problem is solved by the Kropina metric [154]. In this respect, we mention the following result (see [127, Lemma 3.1], [61, Proposition 2.14], [71, Lemma 1.4.1]).

Proposition 6.1.1. *Let (M, F) be a Finsler manifold and W a vector field on M such that $F(x, -W) < 1$. Then the solution of the Zermelo's navigation problem with the navigation data (F, W) is a Finsler metric $\tilde{F}$ obtained by solving the equation*

$$F(x, y - \tilde{F}(x,y)W) = \tilde{F}(x,y), \tag{6.3}$$

for any nonzero $y \in T_xM$, $x \in M$.

Since the indicatrix I_F is strongly convex and it is assumed that $F(x, -W) < 1$, (6.3) admits a unique positive solution $\tilde{F}$ for any nonzero $y \in T_xM$ [61, 127]. Another key remark regarding the inequality $F(x, -W) < 1$ is that it assures the fact that $\tilde{F}$ is a Finsler metric, having the indicatrix $I_{\tilde{F}} = \left\{(x,y) \in TM \mid \tilde{F}(x,y) = 1\right\}$ strongly convex, as well as, for any $x \in M$, $y = 0$ belongs to the region bounded by $I_{\tilde{F}}$; for more details, see [61]. Additionally, any regular piecewise C^∞-curve $\gamma : [0,1] \to M$, parametrized by time, that represents a trajectory in Zermelo's navigation problem has unit $\tilde{F}$-length, i.e. $\tilde{F}(\gamma(t), \dot{\gamma}(t)) = 1$, where $\dot{\gamma}(t)$ is the velocity vector [71, Lemma 1.4.1].

6.2 General (α, β)-metrics

Various examples of Finsler manifolds can be found in the literature and a few of them are outlined in the sequel. Let $\alpha^2 = a_{ij}(x)y^i y^j$ be a quadratic form, where $a_{ij}(x)$ is a Riemannian metric on M, and $\beta = b_i(x)dx^i$ be a differential 1-form on M, also expressed as $\beta = b_i y^i$. The pair (M, F) is called Finsler manifold with *general (α, β)-metric* if the Finsler metric F can be read as $F = \alpha\phi(b^2, s)$, where $\phi(b^2, s)$ is a positive C^∞-function in the variables $b^2 = ||\beta||_\alpha^2 = a^{ij}b_i b_j$ and $s = \frac{\beta}{\alpha}$, with $|s| \leq b < b_0$ and $0 < b_0 \leq \infty$; for more details, see [155].

The examples of general (α,β)-metrics are provided by the slippery slope, slippery-cross-slope metrics and $(\eta,\tilde{\eta})$-slope metric, which have been presented recently in [10, 11, 12, 13]. In the case where ϕ depends only on the variable s, the function $F=\alpha\phi(s)$ is known as (α,β)-*metric.* Such an example is the Randers metric $F=\alpha+\beta$, with $\phi(s)=1+s$ which solves Zermelo's navigation problem under the influence of a weak wind, i.e. $|s|\leq b<1$ [71]. Another example is provided by the Matsumoto metric $F=\frac{\alpha^2}{\alpha-\beta}$, with $\phi(s)=\frac{1}{1-s}$ and $|s|\leq b<\frac{1}{2}$, which carries out the solution to Matsumoto's slope-of-a-mountain problem [106].

From the theory of the general (α,β)-metrics we recall only a few key results for our arguments.

Proposition 6.2.1. *[155] Let M be an n-dimensional manifold. $F=\alpha\phi(b^2,s)$ is a Finsler metric for any Riemannian metric α and 1-form β, with $||\beta||_\alpha<b_0$ if and only if $\phi=\phi(b^2,s)$ is a positive C^∞-function satisfying*

$$\phi-s\phi_2>0,\quad \phi-s\phi_2+(b^2-s^2)\phi_{22}>0,$$

when $n\geq 3$ or

$$\phi-s\phi_2+(b^2-s^2)\phi_{22}>0,$$

when $n=2$, where $s=\frac{\beta}{\alpha}$ and $b=||\beta||_\alpha$ satisfy $|s|\leq b<b_0$.

We notice that ϕ_1 and ϕ_2 denote the derivatives of the function ϕ with respect to the first variable b^2 and the second variable s, respectively. Similarly, ϕ_{12} and ϕ_{22} denote the derivatives of ϕ_1 and ϕ_2 with respect to s. When ϕ is a function only of variable s, the derivatives ϕ_2 and ϕ_{22} are simply denoted by ϕ' and ϕ'', respectively.

To conclude the presentation of the desired results, we also need to recall the following notations

$$r_{ij}=\frac{1}{2}(b_{i|j}+b_{j|i}),\quad r_i=b^jr_{ji},\quad r^i=a^{ij}r_j,\quad r_{00}=r_{ij}y^iy^j,\quad r_0=r_iy^i,\quad r=b^ir_i,$$

$$s_{ij}=\frac{1}{2}(b_{i|j}-b_{j|i}),\quad s_i=b^js_{ji},\quad s^i=a^{ij}s_j,\quad s^i_0=a^{ij}s_{jk}y^k,\quad s_0=s_iy^i, \tag{6.4}$$

with $b^j=a^{ji}b_i$, $b_{i|j}=\frac{\partial b_i}{\partial x^j}-\Gamma^k_{ij}b_k$ and $\Gamma^k_{ij}=\frac{1}{2}a^{km}\left(\frac{\partial a_{jm}}{\partial x^i}+\frac{\partial a_{im}}{\partial x^j}-\frac{\partial a_{ij}}{\partial x^m}\right)$ being the Christoffel symbols of the Riemannian metric a_{ij}. We point out that the differential 1-form β is closed if and only if $s_{ij}=0$ (see [71]).

Proposition 6.2.2. *[155] For a general (α,β)-metric $F=\alpha\phi(b^2,s)$, its spray coefficients $\mathcal{G}^i$ are related to the spray coefficients $\mathcal{G}^i_\alpha$ of α by*

$$\begin{aligned}\mathcal{G}^i &= \mathcal{G}^i_\alpha+\alpha Qs^i_0+\left[\Theta(-2\alpha Qs_0+r_{00}+2\alpha^2Rr)+\alpha\Omega(r_0+s_0)\right]\frac{y^i}{\alpha}\\ &\quad+\left[\Psi(-2\alpha Qs_0+r_{00}+2\alpha^2Rr)+\alpha\Pi(r_0+s_0)\right]b^i-\alpha^2R(r^i+s^i),\end{aligned}$$

where

$$Q=\frac{\phi_2}{\phi-s\phi_2},\qquad \Theta=\frac{(\phi-s\phi_2)\phi_2-s\phi\phi_{22}}{2\phi[\phi-s\phi_2+(b^2-s^2)\phi_{22}]},$$

$$\Psi=\frac{\phi_{22}}{2[\phi-s\phi_2+(b^2-s^2)\phi_{22}]},\qquad \Pi=\frac{(\phi-s\phi_2)\phi_{12}-s\phi_1\phi_{22}}{(\phi-s\phi_2)[\phi-s\phi_2+(b^2-s^2)\phi_{22}]},$$

$$\Omega=\frac{2\phi_1}{\phi}-\frac{s\phi+(b^2-s^2)\phi_2}{\phi}\Pi,\qquad R=\frac{\phi_1}{\phi-s\phi_2}.$$

Chapter 7

Time geodesics on a slippery slope under gravitational wind

The current chapter, which is based on the paper [10], presents the concept of a slippery mountain slope and the purely geometric solutions for time-minimal navigation on such a slope by means of Finsler geometry. This approach allowed us to generalize and interlink Matsumoto's slope-of-a-mountain problem with Zermelo's navigation problem on Riemannian manifolds under the influence of a weak gravitational wind.

7.1 Slippery slope model

Prior to stating the navigation problem on a slippery slope and formulating the main results, it is necessary to introduce some basic concepts, to set up a new terminology and notation.

7.1.1 Gravitational wind

Let (M, h) be a surface embedded in $\mathbb{R}^3$, i.e. a 2-dimensional Riemannian manifold, and π_O be the tangent plane to M at an arbitrary point $O \in M$. Considering that $\mathbf{G}$ is a gravitational field in $\mathbb{R}^3$ that affects a mountain slope M, this can be decomposed into two orthogonal components, $\mathbf{G} = \mathbf{G}^T + \mathbf{G}^\perp$, where $\mathbf{G}^\perp$ is orthogonal and $\mathbf{G}^T$ is tangent to M in O. The latter acts along an anti-gradient (a negative gradient), i.e. the steepest descent (downhill) direction. Further on, $\mathbf{G}^T$ will be called a *gravitational wind*, and the norm of $\mathbf{G}^T$ with respect to h is $||\mathbf{G}^T||_h = \sqrt{h(\mathbf{G}^T, \mathbf{G}^T)}$. In general, $\mathbf{G}^T$ depends on the gradient vector field related to the slope M and a given acceleration of gravity. Note that we set up this terminology in order to be later on in line with the standard nomenclature widely used in different studies on the Zermelo navigation in Finsler geometry [45, 124, 154, 61].

Matsumoto formulated and solved the *slope-of-a-mountain problem* on the surface (M, h) in 1989 by a purely geometric approach in Finsler geometry [106]. By constructing a rectangular basis $\{e_1, e_2\}$ in the tangent plane π_O, where e_1 has the same direction with $\mathbf{G}^T$, he considered a person who walks or runs on π_O expressed by the coordinates X, Y with respect to $\{e_1, e_2\}$ in the clockwise direction θ and with a constant self-speed $||u||_h = a$. Thus, the distance range reached in all possible directions in unit time, i.e. the indicatrix is represented by a limaçon described in the polar coordinates (r, θ) of the plane π_O as

$$r = a + w \cos\theta^1, \tag{7.1}$$

where $w = \frac{g}{2}\sin\varepsilon$, g is the gravitational acceleration, ε is the angle of inclination of π_O with respect to the horizontal plane $z = 0$, and $\sin\varepsilon$ is the norm of the surface gradient with respect to h. The convexity of the limaçon (7.1) is assured under the condition $a > 2w$ [63, 131, 106].

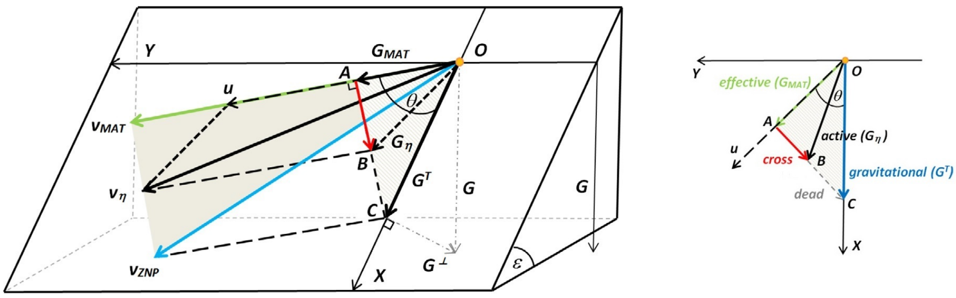


Figure 7.1: A model of a planar slippery slope (an inclined plane of the slope angle ε) under the gravity field, $\mathbf{G} = \mathbf{G}^T + \mathbf{G}^\perp$, normal to the horizontal plane (base of the slope), where $\mathbf{G}^T$ is the gravitational wind acting tangentially to the slope in the steepest downhill direction X and $\mathbf{G}^\perp$ is the component of gravity normal to the slope; $OX \perp OY$. On further reading, the boundary cases represented by the resultant velocities v_{ZNP} (blue), v_{MAT} (green) are denoted by v_η, where $\eta = 0, 1$, i.e. the tangent vectors to the Zermelo-Randers and Matsumoto geodesics, respectively. The lateral component $\overrightarrow{AB}$ (a cross wind; red) of the active wind $\mathbf{G}_\eta = \overrightarrow{OB}$ depends in particular on the traction coefficient η.

It is worth pointing out that our approach presented in [10] substantially extends the original Matsumoto's reasoning. Namely, while the earth's gravity acts on a runner on the slope, the component of a gravitational wind perpendicular to a desired direction of motion represented by a control vector u (cross-gravity effect, i.e. the transversal component of the gravitational wind) is not regarded to be cancelled now, i.e. $||\overrightarrow{AB}||_h \geq 0$ (see Figure 7.1). Obviously, another component $\overrightarrow{OA}$ pushes a runner downhill continuously. In this new setting the slide caused by gravity is also considered off the planned track, that is, not only the gravity additive along the self-velocity u (along-gravity effect, i.e. the longitudinal component of the gravitational wind). As a consequence, the resulting velocity and the self-velocity are not collinear in general, unlike the situation considered in [106], where the cross-gravity effect is omitted[2] or, in other words, completely compensated, i.e. $B = A$. The velocities are collinear, if the steepest route is followed, i.e. the gradient (uphill) or anti-gradient (downhill) direction.

The proposed model refers to a slippery slope of a hill or a mountain in real world, admitting the cross-track slides, the range of which depends on, among others, the interaction between the type of ground on a slope and tread, e.g., shoes, tyres, skis. On the other hand, observe that if the cross-component $\overrightarrow{AB}$ of the gravitational wind is not compensated at all, i.e. $B = C$, then such setting becomes a scenario like in the Zermelo navigation (with the wind navigation data $W = \mathbf{G}^T$), which has been intensively investigated and widely used

[1]The original notation in Matsumoto's paper ([106]) is: $r = v + a\cos\theta$, where $a = w\sin\alpha$.

[2]Matsumoto justifies this issue in a word, assuming that the component perpendicular to u is regarded to be cancelled by planting runner's legs on the desired road determined by u [106, p. 19].

as the efficient geometric method in Finsler geometry over the last twenty years; see, e.g. [45, 76, 127]. This case models a full slide or an undisturbed drift (without any loss of free wind effect) like perfect sailing or flying in water or air currents.

As already mentioned, on the mountainsides in real world situations some specific or external circumstances, e.g. type of ground (wet, muddy, icy) or the tread can cause the slope to be slippery. Due to this fact, we introduce a *cross-traction coefficient* $\eta \in [0,1]$, which has an impact on the sliding effect on a slippery slope being represented by the norm $||\overrightarrow{AB}||_h$ in the plane π_O; for clarity, see Figure 7.1. The lesser the traction, the greater the sliding. Roughly speaking, this parameter generally refers to the outcome of sliding, and not to its causes. In its usual sense, traction means the ability, for instance, of a wheel, tyre or shoe to hold the ground without sliding. Here, due to traction, the gravitational wind acting on the slope can now be written as $\mathbf{G}^T = \overrightarrow{OB} + \overrightarrow{BC}$. The vector $\overrightarrow{OB}$ is called an *active wind*, denoted by $\mathbf{G}_\eta$, and the vector $\overrightarrow{BC}$ is called a *dead wind*, since the former is the actual impact of gravity on the motion and the latter represents the offset (vanished) effect of gravity on the slope due to the existing traction. In short, the norm of dead wind is a measure of compensation of the acting gravitational wind. One can say that the influence of the gravitational wind $\mathbf{G}^T$ "blowing" on a slope is weakened because of η-effect, if we do not follow the steepest routes. Furthermore, the active wind is decomposed as

$$\mathbf{G}_\eta = \overrightarrow{OA} + \overrightarrow{AB}, \tag{7.2}$$

where $\overrightarrow{OA}$ is the orthogonal projection of $\mathbf{G}^T$ (or $\mathbf{G}_\eta$) on the self-velocity u, and it is called an *effective wind*, denoted by $\mathbf{G}_{MAT}$. Thus, $||\mathbf{G}_\eta||_h \in \left[||\mathbf{G}_{MAT}||_h, ||\mathbf{G}^T||_h\right]$, for any $\eta \in [0,1]$ and $||\mathbf{G}_{MAT}||_h \in \left[0, ||\mathbf{G}^T||_h\right]$. Moreover, we call the vector $\overrightarrow{AB}$ a *cross wind*. Its norm is the sliding measure on a slippery slope, generally depending on the traction coefficient η, direction of motion θ and gravitational wind force $||\mathbf{G}^T||_h$.

This movement can be compared to the vessel's sideways sliding motion (called sway) on a dynamic surface of a sea. For the convenience of the reader and ease of presentation, the decompositions of a gravitational wind as well as the corresponding terminology introduced above are shown in Figure 7.1, right.

In particular, if $\eta = 0$, then the slide reaches a maximum ($B = C$) and consequently, the active wind is also maximal, i.e. $\mathbf{G}_\eta = \mathbf{G}^T$. Namely, $\mathbf{G}^T = \overrightarrow{OA} + \overrightarrow{AB}$ and the dead wind vanishes. There is a maximal slide (hence, a maximal drift as well) in this case[3]. After having paid a little thought, as already mentioned, we can see that this scenario leads directly to the navigation problem of Zermelo. If $\eta \neq 0$, then the cross wind becomes shorter, since $B \neq C$ any more, so that the slide effect on a slope is now smaller. On the other hand, if $\eta = 1$, then $B = A$ and the effect of the active wind is minimal (for a given θ and $\mathbf{G}^T$), i.e. $\mathbf{G}_\eta = \mathbf{G}_{MAT}$. Therefore, there is no sliding (and no drift, either) at all, like in the original Matsumoto's setting [106]. One can say that the impact of the gravitational wind $\mathbf{G}^T$ is reduced in the last case as much as possible, since its cross-component is completely compensated. In other words, the dead wind becomes maximal (for a given θ and $\mathbf{G}^T$) then.

Thus, for any $\eta \in [0,1]$ we can write $\overrightarrow{AB} = (1-\eta)\overrightarrow{AC}$ and since $\mathbf{G}^T = \mathbf{G}_{MAT} + \overrightarrow{AC}$, by

[3] A norm of a cross wind, $||\overrightarrow{AB}||_h$, is the linear measure of a slide on the slippery slope, while a drift (a.k.a. a drift angle) is the corresponding angular measure of a slide, i.e. the angle $|\tilde{\theta} - \theta|$ between the self-velocity u and the resultant velocity v_η, where $\tilde{\theta} = \measuredangle\{X, v_\eta\}$ measured clockwise.

(7.2) it results that the active wind can be rephrased as follows

$$\mathbf{G}_\eta = \eta \mathbf{G}_{MAT} + (1-\eta)\mathbf{G}^T. \tag{7.3}$$

Because of the slope being slippery, the self-velocity u is actually perturbed by $\mathbf{G}_\eta$. Hence, the resulting velocity will be given by the composed vector $v_\eta = u + \mathbf{G}_\eta, \quad \eta \in [0,1]$. This general relation defines the equation of motion on a slippery slope.

7.1.2 Main results

Bearing in mind the ones above stated, the *navigation problem on a slippery slope* can be formulated as follows:

> *Suppose a person walks on a horizontal plane at a constant speed, while the gravity acts orthogonally to this plane. Imagine the person endeavours to walk now on the slippery mountainside with a given traction coefficient and under the influence of gravity. How should the person navigate on the slippery slope of a mountain in order to travel from one point to another in the shortest time?*

The main theorem of our work is giving the answer [4] presented below in the general context of an n-dimensional Riemannian manifold with $G^T = -\bar{g}\omega^\sharp$, where $\omega^\sharp$ is the gradient vector field (7.7) and $\bar{g}$ is the rescaled gravitational acceleration g.

Theorem 7.1.1. (Slippery slope metric) *Let a slippery slope of a mountain be an n-dimensional Riemannian manifold (M,h), $n>1$, with the gravitational wind $\mathbf{G}^T$ on M and the cross-traction coefficient $\eta \in [0,1]$. The time-minimal paths on (M,h) in the presence of an active wind $\mathbf{G}_\eta$ as in (7.3) are the geodesics of the slippery slope metric $\tilde{F}_\eta$ which satisfies*

$$\tilde{F}_\eta \sqrt{\alpha^2 + 2(1-\eta)\bar{g}\beta\tilde{F}_\eta + (1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}_\eta^2} = \alpha^2 + (2-\eta)\bar{g}\beta\tilde{F}_\eta + (1-\eta)||\mathbf{G}^T||_h^2\tilde{F}_\eta^2, \tag{7.4}$$

with $\alpha = \alpha(x,y)$, $\beta = \beta(x,y)$ given by (7.14), where either $\eta \in [0,\frac{1}{2}]$ and $||\mathbf{G}^T||_h < 1$, or $\eta \in (\frac{1}{2},1]$ and $||\mathbf{G}^T||_h < \frac{1}{2\eta}$. In particular, if $\eta = 1$, then the slippery slope metric is reduced to the Matsumoto metric, and if $\eta = 0$, then it is the Randers metric which solves the Zermelo navigation problem on a Riemannian manifold under a gravitational wind $\mathbf{G}^T$.

It is worth noting that the metric $\tilde{F}_\eta$ belongs to the class of the general (α,β)-metrics [155] and it stands for a natural and actual model of Finsler spaces, as well as for a new application of this type of Finsler metrics.

In order to solve the problem stated above completely, we hereinafter aim at finding the corresponding (local) time-minimal paths, which are the geodesics of the slippery slope metric. With (7.26) we can determine all such geodesics as follows.

Theorem 7.1.2. (Time geodesics) *Let a slippery slope of a mountain be an n-dimensional Riemannian manifold (M,h), $n>1$, with the gravitational wind $\mathbf{G}^T$ on M and the cross-traction coefficient $\eta \in [0,1]$. The time-minimal paths on (M,h) in the presence of an active wind $\mathbf{G}_\eta$ as in (7.3) are the time-parametrized solutions $\gamma(t) = (\gamma^i(t))$, $i = 1,...,n$ of the ODE system*

$$\ddot{\gamma}^i(t) + 2\tilde{\mathcal{G}}^i_\eta(\gamma(t),\dot{\gamma}(t)) = 0, \tag{7.5}$$

[4] The local solution is given ultimately by a time-minimal trajectory (time geodesic) or, equivalently, by the corresponding direction of self-velocity (optimal control) as a function of time.

where

$$\tilde{\mathcal{G}}^i_\eta(\gamma(t),\dot\gamma(t)) = \mathcal{G}^i_\alpha(\gamma(t),\dot\gamma(t)) + \left[\ \tilde\Theta(r_{00}+2\alpha^2\tilde R r)+\alpha\tilde\Omega r_0\right]\frac{\dot\gamma^i(t)}{\alpha} - \left[\ \tilde\Psi(r_{00}+2\alpha^2\tilde R r)+\alpha\tilde\Pi r_0\right]\frac{w^i}{\bar g} - \tilde R {w^i}_{|j}\frac{\alpha^2 w^j}{\bar g^2},$$

with

$$\begin{aligned}
&\mathcal{G}^i_\alpha(\gamma(t),\dot\gamma(t)) = \tfrac14 h^{im}\left(2\tfrac{\partial h_{jm}}{\partial x^k}-\tfrac{\partial h_{jk}}{\partial x^m}\right)\dot\gamma^j(t)\dot\gamma^k(t),\\
&r_{00}=-\tfrac{1}{\bar g}w_{j|k}\dot\gamma^j(t)\dot\gamma^k(t), \qquad r_0=\tfrac{1}{\bar g^2}w_{j|k}\dot\gamma^j(t)w^k, \qquad r=-\tfrac{1}{\bar g^3}w_{j|k}w^jw^k,\\
&\tilde R=\tfrac{(1-\eta)\bar g^2}{2\tilde B\alpha^4}(\tilde B\alpha^2+2\eta), \qquad \tilde\Theta=\tfrac{\bar g^2\alpha(\tilde A\tilde B^2\alpha^2-2\tilde D^2\beta)}{2\tilde E}, \qquad \tilde\Psi=\tfrac{\bar g^2\alpha^2(\tilde A^2\tilde B+2\tilde D^2)}{2\tilde E},\\
&\tilde\Omega=\tfrac{(1-\eta)\bar g^2}{\tilde B\tilde E\alpha^2}[(\tilde B\alpha^2+2\eta)(\bar g^2\tilde B^3\alpha^2+2\tilde D^2||\mathbf{G}^T||_h^2)-4\eta\tilde D(\bar g^2\tilde B\beta+\tilde A||\mathbf{G}^T||_h^2)],\\
&\tilde\Pi=\tfrac{(1-\eta)\bar g^4}{\tilde B\tilde E\alpha^3}[4\eta\tilde C\tilde D\alpha+(\tilde B\alpha^2+2\eta)(\tilde A\tilde B^2\alpha^2-2\tilde D^2\beta)],\\
&\tilde A=-\tfrac{2\bar g}{\alpha^2}\left\{(1-\eta)[1-(2-\eta)||\mathbf{G}^T||_h^2]-(2-\eta)^2\bar g\beta-(2-\eta)\alpha^2\right\},\\
&\tilde B=-\tfrac{2}{\alpha^2}\left\{[1-2(1-\eta)||\mathbf{G}^T||_h^2]-2(2-\eta)\bar g\beta-2\alpha^2\right\}, \qquad \tilde C=\tfrac1\alpha\left(\tilde B\alpha^2+\tilde A\beta\right),\\
&\tilde D=2\tilde A-(2-\eta)\bar g\tilde B, \qquad \tilde E=\bar g^2\tilde B\tilde C^2\alpha^2+(||\mathbf{G}^T||_h^2\alpha^2-\bar g^2\beta^2)(\tilde A^2\tilde B+2\tilde D^2)
\end{aligned} \tag{7.6}$$

and $\alpha=\alpha(\gamma(t),\dot\gamma(t))$, $\beta=\beta(\gamma(t),\dot\gamma(t))$.

We note that the notation $w_{j|k}$ means the covariant derivative of $\mathbf{G}^T$ (written as $\mathbf{G}^T=w^i\frac{\partial}{\partial x^i}$) with respect to h.

The main results are proved in the next section.

7.2 Proofs of the main results

In order to prove Theorem 7.1.1 we proceed in two steps which include a sequence of lemmas. We consider the navigation problem on a slippery slope that is the n-dimensional Riemannian manifold (M,h), $n>1$, under the influence of an active wind $\mathbf{G}_\eta$ given by (7.3).

Let $p: M\to\mathbb{R}$ be a C^∞-function on M. The image of a differential 1-form $\omega=dp=\frac{\partial p}{\partial x^i}dx^i$ by the musical isomorphism $\sharp$ is the *gradient vector field*

$$\omega^\sharp = h^{ji}\frac{\partial p}{\partial x^j}\frac{\partial}{\partial x^i} \tag{7.7}$$

and $||\omega^\sharp||_h^2=h^{ji}\frac{\partial p}{\partial x^i}\frac{\partial p}{\partial x^j}$. Since a gravitational wind $\mathbf{G}^T$ acts along the anti-gradient, we set $\mathbf{G}^T=-\bar g\omega^\sharp$, where $\bar g$ is the rescaled magnitude of the acceleration of gravity g, i.e. $\bar g=\lambda g$, $\lambda>0$. Moreover, $||\omega^\sharp||_h=\frac{1}{\bar g}||\mathbf{G}^T||_h$.

Let u be the self-velocity with the assumption that $||u||_h=1$ as it is usually set up in the theoretical investigations on the Zermelo navigation. This means that the self-speed $||u||_h$ is normalized and thus, it requires the norm of the active wind $\mathbf{G}_\eta$ to be rescaled accordingly, so that they correspond to each other. We mention that the walker's self-speed on a slope and a gravitational acceleration were not normalized in the original work of Matsumoto [106].

Furthermore, by analogy to other studies on the Zermelo navigation (see e.g. [45, 71, 154, 61]), we set similar classification of types of winds with respect to their force. Namely, if $||\mathbf{G}_\eta||_h < 1$ for any $\eta \in [0,1]$, then $\mathbf{G}_\eta$ is *weak*; if $||\mathbf{G}_\eta||_h = 1$ for any $\eta \in [0,1]$, then $\mathbf{G}_\eta$ is *critical,* and if $||\mathbf{G}_\eta||_h > 1$ for any $\eta \in [0,1]$, then $\mathbf{G}_\eta$ is *strong.*

Under the effect of the weak active wind $\mathbf{G}_\eta$, having the resulting velocity $v_\eta = u + \mathbf{G}_\eta$, $\eta \in [0,1]$, it seems that the metric obtained by the Zermelo navigation method is a Finsler metric of Randers type like, for example in [45, 71, 78]. However, as it will be shown below, it is in fact much more complicated because only the gravitational wind $\mathbf{G}^T$ is known (a priori), and the vector $\mathbf{G}_{MAT}$ depends on the direction of the self-velocity, being the orthogonal projection of $\mathbf{G}^T$ on u. Therefore, we conveniently split this proof into two main steps including some cases. First, we deform the background Riemannian metric by the vector field $\eta\mathbf{G}_{MAT}$, which is the direction-dependent deformation. In the second step, the resulting Finsler metric F (obtained in the first step) is deformed by the vector field $(1-\eta)\mathbf{G}^T$, under the condition $F(x, -(1-\eta)\mathbf{G}^T) < 1$ which guarantees that the walker on the slippery mountainside can go forward in any direction.

7.2.1 Step I: direction-dependent deformation

In this step we show that, when $\eta||\mathbf{G}_{MAT}||_h < 1$, the direction-dependent deformation of the background Riemannian metric h by the vector field $\eta\mathbf{G}_{MAT}$ defines a Finsler metric if and only if $||\mathbf{G}^T||_h < \frac{1}{2\eta}$, for any $\eta \in (0,1]$. More precisely, the deformation of the Riemannian metric h by the vector field $\eta\mathbf{G}_{MAT}$ invokes the expression of the resulting velocity, i.e. $v = u + \eta\mathbf{G}_{MAT}$, for any $\eta \in (0,1]$. Notice that if $\eta = 0$, this turns out $v = u$. Otherwise, when $\eta||\mathbf{G}_{MAT}||_h \geq 1$ at some direction, this deformation cannot provide a Finsler metric.

Let θ be the angle between $\mathbf{G}^T$ and u, in other words, it represents the desired direction of motion. Since $\mathbf{G}_{MAT} = \mathrm{Proj}_u\mathbf{G}^T$, the vectors v, u and $\mathbf{G}_{MAT}$ are collinear. It follows that the angle between u and $\mathbf{G}_{MAT}$, denoted by $\bar{\theta}$ is 0 or π, or it is not determined, if θ is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ (i.e. u and $\mathbf{G}^T$ are orthogonal and $\mathbf{G}_{MAT} = 0$ in that event).

Taking into account all possibilities for the force of $\eta\mathbf{G}_{MAT}$, we analyze the following cases: 1. $||\eta\mathbf{G}_{MAT}||_h < 1$; 2. $||\eta\mathbf{G}_{MAT}||_h = 1$; and 3. $||\eta\mathbf{G}_{MAT}||_h > 1$.

Case 1. Due to the condition $||\eta\mathbf{G}_{MAT}||_h < 1$, the angle between $\mathbf{G}^T$ and v is also θ (the vectors u and v point to the same direction). Regarding of $\bar{\theta}$, we study two subcases:
i) First, if $\bar{\theta} = 0$ (going downhill), then $\theta \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$, and the angle between $\mathbf{G}^T$ and $\mathbf{G}_{MAT}$ is θ or $2\pi - \theta$. It results that $||\mathbf{G}_{MAT}||_h = ||\mathbf{G}^T||_h \cos\theta$ and also,

$$h(v, \mathbf{G}_{MAT}) = ||v||_h||\mathbf{G}_{MAT}||_h = ||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T).$$

Furthermore, we have $\eta||\mathbf{G}^T||_h \cos\theta < 1$ and $\frac{h(v,\mathbf{G}^T)}{||v||_h} < \frac{1}{\eta}$, $\eta \in (0,1]$.
ii) Second, if $\bar{\theta} = \pi$ (going uphill), then $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$. This means that the angle between $\mathbf{G}^T$ and $\mathbf{G}_{MAT}$ is $\theta - \pi$ or $\pi - \theta$, and it implies that $||\mathbf{G}_{MAT}||_h = -||\mathbf{G}^T||_h \cos\theta$ and

$$h(v, \mathbf{G}_{MAT}) = -||v||_h||\mathbf{G}_{MAT}||_h = ||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T).$$

It follows that $-\eta||\mathbf{G}^T||_h \cos\theta < 1$ and $-\frac{h(v,\mathbf{G}^T)}{||v||_h} < \frac{1}{\eta}$, $\eta \in (0,1]$.

To sum up, by both above possibilities and noting that $v = u$ if $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, we have obtained

$$h(v, \mathbf{G}_{MAT}) = h(v, \mathbf{G}^T) = ||v||_h||\mathbf{G}^T||_h \cos\theta, \text{ for any } \theta \in [0, 2\pi), \tag{7.8}$$

and the condition $||\eta\mathbf{G}_{MAT}||_h < 1$ is equivalent to

$$\eta||\mathbf{G}^T||_h|\cos\theta| < 1 \quad \text{ or } \quad \frac{|h(v, \mathbf{G}^T)|}{||v||_h} < \frac{1}{\eta}, \tag{7.9}$$

for any $\eta \in (0, 1]$ and $\theta \in [0, 2\pi)$. We note that, due to the condition $\eta||\mathbf{G}_{MAT}||_h < 1$, there is not any direction where the resultant vector v vanishes. Now, taking into account (7.8) and (7.9), the relation $1 = ||u||_h = ||v - \eta\mathbf{G}_{MAT}||_h$ leads to

$$||v||_h^2 - 2\eta||v||_h||\mathbf{G}^T||_h \cos\theta - (1 - \eta^2||\mathbf{G}^T||_h^2 \cos^2\theta) = 0.$$

This admits only one positive root

$$||v||_h = 1 + \eta||\mathbf{G}^T||_h \cos\theta, \text{ for any } \theta \in [0, 2\pi). \tag{7.10}$$

Using again (7.8), it can be rewritten as $g_1(x, v) = 0$, where

$$g_1(x, v) = ||v||_h^2 - ||v||_h - \eta h(v, \mathbf{G}^T). \tag{7.11}$$

Applying Okubo's method [106], we obtain the function $F(x, v) = \frac{||v||_h^2}{||v||_h+\eta h(v,\mathbf{G}^T)}$ as the solution of the equation $g_1(x, \frac{v}{F}) = 0$. The extension of $F(x, v)$ to an arbitrary nonzero vector $y \in T_xM$, for any $x \in M$, is the positive homogeneous C^∞-function on TM_0

$$F(x, y) = \frac{||y||_h^2}{||y||_h + \eta h(y, \mathbf{G}^T)}, \text{ for any } \eta \in (0, 1]. \tag{7.12}$$

This is because any nonzero y can be expressed as $y = cv$, $c > 0$, and $F(x, v) = 1$.

Actually, we are able to prove that $\eta||\mathbf{G}_{MAT}||_h < 1$ is a neccesary and sufficient condition for $F(x, y)$, obtained in (7.12), to be positive on all TM_0. Indeed, the positivity of (7.12) on TM_0 means that

$$||y||_h + \eta h(y, \mathbf{G}^T) > 0, \tag{7.13}$$

for all nonzero y and any $\eta \in (0, 1]$. If the positivity holds on TM_0, we can substitute y with $-\mathbf{G}^T \neq 0$ into (7.13) and thus, $\eta||\mathbf{G}^T||_h < 1$. Since $||\mathbf{G}_{MAT}||_h \leq ||\mathbf{G}^T||_h$ in any direction ($||\mathbf{G}_{MAT}||_h = ||\mathbf{G}^T||_h|\cos\theta|$, for any $\theta \in [0, 2\pi)$) it turns out that $\eta||\mathbf{G}_{MAT}||_h < 1$ on all TM_0. Conversely, suppose that $\eta||\mathbf{G}_{MAT}||_h < 1$ on all TM_0 (the case $\mathbf{G}_{MAT} = 0$ is also included). Using (7.9), it follows that $\frac{|h(y,\mathbf{G}^T)|}{||y||_h} < \frac{1}{\eta}$ for any nonzero y, which assures (7.13). Thus, $F(x, y)$ is positive on TM_0.

With the notation

$$\alpha^2 = ||y||_h^2 = h_{ij}y^iy^j \quad \text{ and } \quad \beta = -\frac{1}{\bar{g}}h(y, \mathbf{G}^T) = h(y, \omega^\sharp) = b_iy^i, \tag{7.14}$$

$\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ and $||\beta||_h = ||\omega^\sharp||_h$, the function (7.12) is of Matsumoto type, namely

$$F(x, y) = \frac{\alpha^2}{\alpha - \eta\bar{g}\beta}, \text{ for any } \eta \in (0, 1], \tag{7.15}$$

with the corresponding indicatrix $I_F = \left\{(x,y) \in TM_0 \mid \alpha^2(\alpha - \eta\bar{g}\beta)^{-1} = 1\right\}$. Moreover, $F(x,y)$ can be extended continuously to all TM, i.e. $F(x,0) = 0$ for any $x \in M$, because $y = 0$ does not lie in the closure in TM of the indicatrix I_F [61].

In order to establish the necessary and sufficient conditions for the function (7.15) to be a Finsler metric for any $\eta \in (0,1]$, let us write $F(x,y) = \alpha\phi(s)$, where $\phi(s) = \frac{1}{1-\eta\bar{g}s}$ and $s = \frac{\beta}{\alpha}$. Since the second inequality in (7.9) can be read as $|s| < \frac{1}{\eta\bar{g}}$, for arbitrary nonzero $y \in T_xM$ and $x \in M$, it turns out that ϕ is a positive C^∞-function in this case.

Now, we are able to prove some additional properties regarding ϕ and to control the force of the gravitational wind $\mathbf{G}^T$ via the variable s. More precisely, we have

Lemma 7.2.1. *For any $\eta \in (0,1]$, fhe following statements are equivalent:*
i) $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, where $b = ||\omega^\sharp||_h$;
ii) $|s| \leq b < b_0$, where $b_0 = \frac{1}{2\eta\bar{g}}$;
iii) $||\mathbf{G}^T||_h < \frac{1}{2\eta}$.

Proof. i) $\Leftrightarrow$ ii). Applying the Cauchy-Schwarz inequality $|h(y,\omega^\sharp)| \leq ||y||_h||\omega^\sharp||_h$, it yields $|s| \leq ||\omega^\sharp||_h$ and thus, $|s| \leq b$. Since $|s| < \frac{1}{\eta\bar{g}}$, we have

$$(b^2 - s^2)\phi''(s) = (b^2 - s^2)\frac{2\eta^2\bar{g}^2}{(1-\eta\bar{g}s)^3} \geq 0$$

and the minimum value of $(b^2 - s^2)\phi''(s)$ is achieved when $|s| = b$.

Therefore, if $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, then for $s = b$ it yields $1 - 2\eta\bar{g}b > 0$, so $b < \frac{1}{2\eta\bar{g}}$. Thus, $|s| \leq b < \frac{1}{2\eta\bar{g}}$.

Conversely, if $|s| \leq b < \frac{1}{2\eta\bar{g}}$, then

$$\begin{aligned}\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) &= \frac{(1-\eta\bar{g}s)(1-2\eta\bar{g}s) + 2\eta^2\bar{g}^2(b^2 - s^2)}{(1-\eta\bar{g}s)^3}\\ &\geq \frac{(1-\eta\bar{g}s)(1-2\eta\bar{g}s)}{(1-\eta\bar{g}s)^3} = \frac{1-2\eta\bar{g}s}{(1-\eta\bar{g}s)^2} > 0.\end{aligned}$$

ii) $\Leftrightarrow$ iii). If $||\mathbf{G}^T||_h < \frac{1}{2\eta}$ and making use of the inequality $|s| \leq ||\omega^\sharp||_h$ and $||\omega^\sharp||_h = \frac{1}{\bar{g}}||\mathbf{G}^T||_h$, it implies that $|s| \leq \frac{1}{\bar{g}}||\mathbf{G}^T||_h < \frac{1}{2\eta\bar{g}}$. The converse implication is immediate from $b = \frac{1}{\bar{g}}||\mathbf{G}^T||_h$. □

We note that the statement $|s| \leq b < \frac{1}{2\eta\bar{g}}$, for any $\eta \in (0,1]$ also implies $\phi(s) - s\phi'(s) > 0$.

Next, applying [71, Lemma 1.1.2] and Proposition 6.2.1 we have proved the following result.

Lemma 7.2.2. *For any $\eta \in (0,1]$, the following statements are equivalent:*
i) $F(x,y) = \frac{\alpha^2}{\alpha - \eta\bar{g}\beta}$ is a Finsler metric;
ii) $||\mathbf{G}^T||_h < \frac{1}{2\eta}$.

In summary, we emphasize that the indicatrix I_F is strongly convex if and only if $||\mathbf{G}^T||_h < \frac{1}{2\eta}$, for any $\eta \in (0,1]$.

Case 2. Now we assume that $||\eta \mathbf{G}_{MAT}||_h = 1$. We notice that a traverse of a mountain, i.e. $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ cannot be followed here because it gives $\eta||\mathbf{G}_{MAT}|| = 0$, which contradicts our assumption.

Due to the fact that $||\mathbf{G}_{MAT}||_h \leq ||\mathbf{G}^T||_h$, it turns out that $||\mathbf{G}^T||_h \geq \frac{1}{\eta}$, for any $\eta \in (0, 1]$. Again we have to analyse both possibilities for $\bar{\theta}$:
i) First, if $\bar{\theta} = 0$, then $\theta \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$ and $u = \eta \mathbf{G}_{MAT}$. The vectors u and v have the same direction and $\measuredangle(\mathbf{G}^T, \mathbf{G}_{MAT}) \in \{\theta, 2\pi - \theta\}$. These lead to $||\mathbf{G}_{MAT}||_h = ||\mathbf{G}^T||_h \cos\theta$. Furthermore,

$$h(v, \mathbf{G}_{MAT}) = ||v||_h||\mathbf{G}_{MAT}||_h = ||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T).$$

Since $v = u + \eta \mathbf{G}_{MAT}$, then $v = 2\eta \mathbf{G}_{MAT}$ and thus, $||v||_h = 2$. Moreover, also it turns out that $\cos\theta = \frac{1}{\eta||\mathbf{G}^T||_h}$.
ii) Second, if $\bar{\theta} = \pi$, then $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $u = -\eta \mathbf{G}_{MAT}$, so $||v||_h = 1 - ||\eta \mathbf{G}_{MAT}||_h = 0$. Thus, the resultant velocity v vanishes, while attempting to climb up the slope.

Summing up the above findings, when v does not vanish, we have $||v||_h = 2$ and among the directions corresponding to $\theta \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$ only such directions for which $\cos\theta = \frac{1}{\eta||\mathbf{G}^T||_h}$, i.e. $\frac{h(v, \mathbf{G}^T)}{||v||_h} = \frac{1}{\eta}$, can be followed in this case. Let us consider $g_2(x, v) = 0$, where $g_2(x, v) = ||v||_h - 2$. By Okubo's method [106], we get the function

$$F(x, v) = \frac{1}{2}||v||_h \tag{7.16}$$

as the solution of the equation $g_2(x, \frac{v}{F}) = 0$.

The extension of $F(x, v)$ to an arbitrary non-zero vector $y \in \mathcal{A}_x = \mathcal{A} \cap T_xM$, for any $x \in M$, is $F(x, y) = \frac{1}{2}||y||_h$, where $\mathcal{A} = \{(x, y) \in TM_0 \mid ||y||_h - \eta h(y, \mathbf{G}^T) = 0\}$ is an open conic subset of TM_0. Since $\mathbf{G}^T \neq 0$ and $||\mathbf{G}^T||_h \geq \frac{1}{\eta}$ it results that $c\mathbf{G}^T \in \mathcal{A}_x$, $c > 0$, if and only if $\mathbf{G}^T = \mathbf{G}_{MAT}$. Indeed, if we substitute y with $c\mathbf{G}^T$ into $||y||_h - \eta h(y, \mathbf{G}^T) = 0$, we get $\eta||\mathbf{G}^T||_h = 1$ and thus, the angle θ can be only 0 and $\mathbf{G}^T = \mathbf{G}_{MAT} = \mathbf{G}_\eta$, for any $\eta \in (0, 1]$. Concluding, the function (7.16) can be treated as a conic Finsler metric on $\mathcal{A}$ which is homothetic with the background Riemannian metric h on $\mathcal{A}$ [61, 87]. Anyway, this case does not provide a Finsler metric.

Case 3. The condition $||\eta \mathbf{G}_{MAT}||_h > 1$ implies that the resultant velocity vector v and $\mathbf{G}_{MAT}$ point to the same (downhill) direction irrespective of $\bar{\theta}$. It also gives $||\mathbf{G}^T||_h > \frac{1}{\eta}$. As above we split our investigation into some subcases:
i) First, under the assumption $\bar{\theta} = 0$ it follows that $\theta \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$ and $\measuredangle(\mathbf{G}^T, v) = \measuredangle(\mathbf{G}^T, \mathbf{G}_{MAT}) \in \{\theta, 2\pi - \theta\}$. Accordingly, it follows that $||\mathbf{G}_{MAT}||_h = ||\mathbf{G}^T||_h \cos\theta$ and

$$h(v, \mathbf{G}_{MAT}) = ||v||_h||\mathbf{G}_{MAT}||_h = ||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T).$$

Furthermore, we have $\eta||\mathbf{G}^T||_h \cos\theta > 1$ and $\frac{h(v, \mathbf{G}^T)}{||v||_h} > \frac{1}{\eta}$, for any $\eta \in (0, 1]$.
ii) Second, if $\bar{\theta} = \pi$, then $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\measuredangle(\mathbf{G}^T, v) = \measuredangle(\mathbf{G}^T, \mathbf{G}_{MAT}) = |\theta - \pi|$. In consequence, $||\mathbf{G}_{MAT}||_h = -||\mathbf{G}^T||_h \cos\theta$ and

$$h(v, \mathbf{G}_{MAT}) = ||v||_h||\mathbf{G}_{MAT}||_h = -||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T).$$

Moreover, it yields $-\eta||\mathbf{G}^T||_h \cos\theta > 1$ and $\frac{h(v, \mathbf{G}^T)}{||v||_h} > \frac{1}{\eta}$, for any $\eta \in (0, 1]$.

Finally, if $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, then $v = u$. This yields $\eta||G_{MAT}||_h = 0$, which is contrary to our assumption. To sum up, by both of the above subcases and since G_{MAT} cannot be vanished, we obtain

$$h(v, \mathbf{G}_{MAT}) = h(v, \mathbf{G}^T) = ||v||_h||\mathbf{G}^T||_h|\cos\theta|, \text{ for any } \theta \in [0, 2\pi) \smallsetminus \{\pi/2, 3\pi/2\}, \tag{7.17}$$

and the assumption $||\eta\mathbf{G}_{MAT}||_h > 1$ is equivalent to

$$|\cos\theta| > \frac{1}{\eta||\mathbf{G}^T||_h} \quad \text{or} \quad \frac{h(v, \mathbf{G}^T)}{||v||_h} > \frac{1}{\eta}, \tag{7.18}$$

for any $\eta \in (0, 1]$. Thus, among the directions corresponding to $\theta \in [0, 2\pi) \smallsetminus \{\pi/2, 3\pi/2\}$ only such directions for which $|\cos\theta| > \frac{1}{\eta||\mathbf{G}^T||_h}$ can be followed in this case. Also, $\eta||\mathbf{G}_{MAT}||_h > 1$ attests that there is not any direction where the resultant vector v vanishes.

Now using (7.17), (7.18) and $1 = ||u||_h = ||v - \eta\mathbf{G}_{MAT}||_h$, we are led to

$$||v||_h^2 - 2\eta||v||_h||\mathbf{G}^T||_h|\cos\theta| - (1 - \eta^2||\mathbf{G}^T||_h^2\cos^2\theta) = 0.$$

This admits two positive roots

$$||v||_h = \pm 1 + \eta||\mathbf{G}^T||_h|\cos\theta| \tag{7.19}$$

due to the assumed condition $|\cos\theta| > \frac{1}{\eta||\mathbf{G}^T||_h}$, for any $\theta \in [0, 2\pi) \smallsetminus \{\pi/2, 3\pi/2\}$. Having (7.17), we can express (7.19) as $g_3(x, v) = 0$, where $g_3(x, v) = ||v||_h^2 \mp ||v||_h - \eta h(v, \mathbf{G}^T)$. Again by Okubo's method [106], the solutions of the equation $g_3(x, \frac{v}{F}) = 0$ are the functions $F_{1,2}(x, v) = \frac{||v||_h^2}{\pm||v||_h + \eta h(v, \mathbf{G}^T)}$. They can be extended to an arbitrary nonzero vector $y \in \mathcal{A}_x^* = \mathcal{A}^* \cap T_xM$, for any $x \in M$, because any such nonzero y can be expressed as $y = cv$, $c > 0$, where

$$\mathcal{A}^* = \{(x, y) \in TM \mid \ ||y||_h - \eta h(y, \mathbf{G}^T) < 0\}$$

is an open conic subset of TM_0, for any $\eta \in (0, 1]$. Note that $\mathbf{G}^T \in \mathcal{A}_x^*$ and $-c\mathbf{G}^T \notin \mathcal{A}_x^*$, with $c > 0$. Therefore, we obtain the positive homogeneous functions

$$F_{1,2}(x, y) = \frac{||y||_h^2}{\pm||y||_h + \eta h(y, \mathbf{G}^T)}, \tag{7.20}$$

on $\mathcal{A}^*$, with $F_{1,2}(x, v) = 1$. Following the notation (7.14), the functions (7.20) are of Matsumoto type, i.e.

$$F_{1,2}(x, y) = \frac{\alpha^2}{\pm\alpha - \eta\bar{g}\beta}. \tag{7.21}$$

However, $F_{1,2}$ can give us at most conic Finsler metrics due to their conic domain $\mathcal{A}^*$, rewritten as $\mathcal{A}^* = \{(x, y) \in TM \mid \alpha + \eta\bar{g}\beta < 0\}$. Applying [87, Corollary 4.15], we obtain that both $F_{1,2}$ are strongly convex on $\mathcal{A}^*$ and thus, they are conic Finsler metrics on $\mathcal{A}^*$, for any $\eta \in (0, 1]$. Indeed, for $F_{1,2}$ the strongly convex conditions $(\alpha \mp 2\eta\bar{g}\beta)(\alpha \mp \eta\bar{g}\beta) > 0$ are satisfied for any $(x, y) \in \mathcal{A}^*$ and $\eta \in (0, 1]$.

Consequently, the direction-dependent deformation of the background Riemannian metric h by the vector field $\eta\mathbf{G}_{MAT}$, restricted to $\eta||\mathbf{G}_{MAT}||_h < 1$ for every direction (which is equivalent to $||\mathbf{G}^T||_h < \frac{1}{\eta}$) provides the Finsler metric of Matsumoto type $F(x, y) = \frac{\alpha^2}{\alpha - \eta\beta}$ if and only if $||\mathbf{G}^T||_h < \frac{1}{2\eta}$, for any $\eta \in (0, 1]$. In dimension 2 this means that the Riemannian indicatrix which is unit time circle with respect to h is deformed into limaçon (the locus of unit time destinations in windy conditions), instead of being rigidly translated as in the Zermelo navigation.

7.2.2 Step II: rigid translation

Taking into consideration [61], the second step concerns the fact that the addition of the gravitational wind $\mathbf{G}^T$ only generates a rigid translation to the indicatrix provided by the equation of motion $v = u + \eta \mathbf{G}_{MAT}$ in the first step. We can discard the case $\eta ||\mathbf{G}_{MAT}||_h \geq 1$ because this gave us only conic Finsler metrics and thus, going forward in any direction is not possible. Moreover, the translations of the resulting conic Finsler metrics (from *Cases 2* and *3*) may not exist for any $\eta \in (0, 1]$.

Proposition 6.1.1 forms the basis for our subsequent study. Namely, we consider the Zermelo navigation on the Finsler manifold (M, F) with the navigation data $(F, (1-\eta)\mathbf{G}^T)$, for any $\eta \in [0, 1]$ and under the condition

$$F(x, -(1-\eta)\mathbf{G}^T) < 1, \tag{7.22}$$

where F is the Finsler metric (7.15) if $\eta \in (0, 1]$, and $F = h$ if $\eta = 0$. The solution of Zermelo's navigation problem yields the Finsler slippery slope metric $\tilde{F}$ which is the unique positive solution of the equation

$$F(x, y - (1-\eta)\tilde{F}(x, y)\mathbf{G}^T) = \tilde{F}(x, y), \tag{7.23}$$

for any $(x, y) \in TM_0$. In particular, if $\eta = 1$, then $\tilde{F} = \frac{\alpha^2}{\alpha - \bar{g}\beta}$, i.e. F from (7.15) with $\eta = 1$.

Writing the Finsler metric F as $F(x, y) = \frac{\alpha^2}{\alpha - \eta\bar{g}\beta}$, for any $\eta \in [0, 1]$, our next objective is to use it in (7.23) in order to reach the slippery slope metric $\tilde{F}$. A direct computation gives

$$\alpha^2\left(x, y - (1-\eta)\tilde{F}(x, y)\mathbf{G}^T\right) = \alpha^2(x, y) + 2(1-\eta)\bar{g}\beta(x, y)\tilde{F}(x, y) + (1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}^2(x, y)$$

and

$$\beta\left(x, y - (1-\eta)\tilde{F}(x, y)\mathbf{G}^T\right) = \beta(x, y) + (1-\eta)\frac{1}{\bar{g}}||\mathbf{G}^T||_h^2\tilde{F}(x, y),$$

because $\beta(x, \mathbf{G}^T) = -\frac{1}{\bar{g}}||\mathbf{G}^T||_h^2$. Thus, (7.23) leads to the irrational equation

$$\tilde{F}\sqrt{\alpha^2 + 2(1-\eta)\bar{g}\beta\tilde{F} + (1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}^2} = \alpha^2 + (2-\eta)\bar{g}\beta\tilde{F} + (1-\eta)||\mathbf{G}^T||_h^2\tilde{F}^2, \tag{7.24}$$

for any $\tilde{\eta} \in [0, 1]$, where α, β and $\tilde{F}$ are evaluated at (x, y).

From (7.24) we can extract two classic cases that are well known in literature on Finsler geometry. Namely, first, if $\eta = 1$ and $\bar{g} = 1$, then (7.24) yields the standard Matsumoto metric $\tilde{F}(x, y) = \frac{\alpha^2}{\alpha - \beta}$, with $||\mathbf{G}^T||_h < \frac{1}{2}$ and this solves Matsumoto's slope-of-a-mountain problem. Second, if $\eta = 0$, then the gravitational wind $\mathbf{G}^T$ is not compensated at all. More precisely, the dead wind is zeroed or, equivalently, the cross wind is maximal. It has the same nature as a wind included in the standard navigation data of the Zermelo problem, namely, wind force is not reduced, although its direction is fixed, i.e. the steepest descent. Therefore, $\eta = 0$ in (7.24) leads to

$$\tilde{F}\sqrt{\alpha^2 + 2\bar{g}\beta\tilde{F} + ||\mathbf{G}^T||_h^2\tilde{F}^2} = \alpha^2 + 2\bar{g}\beta\tilde{F} + ||\mathbf{G}^T||_h^2\tilde{F}^2. \tag{7.25}$$

Since $\alpha^2 + 2\bar{g}\beta\tilde{F} + ||\mathbf{G}^T||_h^2\tilde{F}^2 > 0$, (7.25) is reduced to

$$(1 - ||\mathbf{G}^T||_h^2)\tilde{F}^2 - 2\bar{g}\beta\tilde{F} - \alpha^2 = 0,$$

which admits only the positive root $\tilde{F}(x,y)=\frac{\sqrt{\alpha^2(1-||\mathbf{G}^T||_h^2)+\bar{g}^2\beta^2}+\bar{g}\beta}{1-||\mathbf{G}^T||_h^2}$, under the weak gravitational wind, $||\mathbf{G}^T||_h<1$. With the notation

$$\tilde{\alpha}^2=\frac{\alpha^2(1-||\mathbf{G}^T||_h^2)+\bar{g}^2\beta^2}{(1-||\mathbf{G}^T||_h^2)^2} \quad \text{and} \quad \tilde{\beta}=\frac{\bar{g}\beta}{1-||\mathbf{G}^T||_h^2},$$

we have $\tilde{F}(x,y)=\tilde{\alpha}+\tilde{\beta}$. This is the Randers metric which solves Zermelo's navigation problem under the weak gravitational wind $\mathbf{G}^T$.

Now coming back to the general case, where $\eta\in[0,1]$, (7.24) is equivalent to the polynomial equation of degree four, that is

$$\begin{aligned}&(1-\eta)^2\,||\mathbf{G}^T||_h^2\left(1-||\mathbf{G}^T||_h^2\right)\tilde{F}^4+2\,(1-\eta)\left[1-(2-\eta)\,||\mathbf{G}^T||_h^2\right]\bar{g}\beta\tilde{F}^3\\&+\left\{\left[1-2\,(1-\eta)\,||\mathbf{G}^T||_h^2\right]\alpha^2-(2-\eta)^2\,\bar{g}^2\beta^2\right\}\tilde{F}^2-2\,(2-\eta)\,\bar{g}\alpha^2\beta\tilde{F}-\alpha^4=0,\end{aligned} \tag{7.26}$$

which admits four roots if $(1-\eta)^2\left(1-||\mathbf{G}^T||_h^2\right)\neq 0$. However, taking into consideration the condition (7.22) (see [61, p. 10 and Proposition 2.14]) it follows that there is a unique positive root, i.e. the slippery slope metric. Subsequently, it is denoted by $\tilde{F}_\eta$ and it satisfies (7.24), for each $\eta\in[0,1]$. We note that along any regular piecewise C^∞-curve γ, parametrized by time that represents a trajectory in Zermelo's problem, $\tilde{F}(\gamma(t),\dot{\gamma}(t))=1$, i.e. the time in which a craft or a vehicle goes along it. We remark that using a computational software system, e.g. Wolfram Mathematica one can generate all four roots of the last equation, but their explicit forms are very complicated.

Now, we pay more attention to the condition (7.22), which assures that the indicatrix of $\tilde{F}_\eta$ (i.e. the unique positive solution of the equation (7.23)) is strongly convex.

Lemma 7.2.3. *The following statements are equivalent:*
i) the indicatrix $I_{\tilde{F}_\eta}$ of the slippery slope metric $\tilde{F}_\eta$ is strongly convex;
ii) the gravitational wind $\mathbf{G}^T$ is weak with either $||\mathbf{G}^T||_h<1$ and $\eta\in[0,\frac{1}{2}]$, or $||\mathbf{G}^T||_h<\frac{1}{2\eta}$ and $\eta\in(\frac{1}{2},1]$;
iii) the active wind $\mathbf{G}_\eta$ given by (7.3) is weak with either $||\mathbf{G}_\eta||_h<1$ and $\eta\in[0,\frac{1}{2}]$, or $||\mathbf{G}_\eta||_h<\frac{1}{2\eta}$ and $\eta\in(\frac{1}{2},1]$.

Proof. i) $\Leftrightarrow$ ii). Having developed the expression

$$F(x,-(1-\eta)\mathbf{G}^T)=\frac{||-(1-\eta)\mathbf{G}^T||_h^2}{||-(1-\eta)\mathbf{G}^T||_h+\eta h(-(1-\eta)\mathbf{G}^T,\mathbf{G}^T)}=\frac{(1-\eta)||\mathbf{G}^T||_h}{1-\eta||\mathbf{G}^T||_h},$$

an elementary calculation shows that (7.22) is equivalent to $||\mathbf{G}^T||_h<1$, for any $\eta\in(0,1)$. Since for $\eta=0$, $F=h$ and then the condition (7.22) also means that $||\mathbf{G}^T||_h<1$.

If we combine the last condition with the strong convexity restriction for the indicatrix I_F (more precisely, with $||\mathbf{G}^T||_h<\frac{1}{2\eta}$), for any $\eta\in(0,1]$, we see that the indicatrix $I_{\tilde{F}_\eta}$ is strongly convex if and only if either $||\mathbf{G}^T||_h<1$ and $\eta\in[0,\frac{1}{2}]$, or $||\mathbf{G}^T||_h<\frac{1}{2\eta}$ and $\eta\in(\frac{1}{2},1]$. Since $\frac{1}{2\eta}<1$, for any $\eta\in(\frac{1}{2},1]$, we outline that the gravitational wind $\mathbf{G}^T$ is weak for any $\eta\in[0,1]$.

ii) $\Leftrightarrow$ iii). The main key to prove that ii) is equivalent to iii) is the remark that $||\mathbf{G}_\eta||_h\leq||\mathbf{G}^T||_h$, for any $\eta\in[0,1]$ and moreover, the maximum of $||\mathbf{G}_\eta||_h$ coincides with $||\mathbf{G}^T||_h$, since $\mathbf{G}_{MAT}$ must vanish for some directions. □

Therefore, as it is emphasized by Lemma 7.2.3, the situation on the slippery slope under weak active wind $\mathbf{G}_\eta$ can be described in the terms of the weak gravitational wind $\mathbf{G}^T$ instead, for any $\eta \in [0,1]$.

Remark that in general the Finslerian geodesics do not minimize time locally, if the related indicatrix is not strongly convex, since the triangle inequality does not hold in such situation.

Summarizing the results obtained in Steps I and II, we have proved Theorem 7.1.1.

7.2.3 Geodesics of the slippery slope metric

This subsection mainly presents the proof of Theorem 7.1.2 which is based on some technical computations which we split in a few lemmas. It is worth pointing out that even if we have not an explicit formula for the slippery slope metric $\tilde{F}_\eta$, for each $\eta \in [0,1]$, we can find the time-minimal paths as the geodesics γ of $\tilde{F}_\eta$, taking into account the fact that $\tilde{F}_\eta(\gamma(t), \dot{\gamma}(t)) = 1$ along them.

First of all, analyzing (7.26), we can conclude that $\tilde{F}_\eta$ depends on the variables $||\mathbf{G}^T||_h$ and $s = \frac{\beta}{\alpha}$, where η being only a parameter. Thus, the slippery slope metric $\tilde{F}_\eta$ is a general (α,β)-metric, $\tilde{F}_\eta(x,y) = \alpha\tilde{\phi}_\eta(||\mathbf{G}^T||_h^2, s)$, since $\bar{g}^2 b^2 = ||\mathbf{G}^T||_h^2$, where $\tilde{\phi}_\eta$ is a positive C^∞-function as well as α and β are given by (7.14).

Furthermore, $\tilde{\phi}_\eta$ is the unique positive solution of the polynomial equation

$$\begin{aligned}&(1-\eta)^2||\mathbf{G}^T||_h^2\left(1-||\mathbf{G}^T||_h^2\right)\tilde{\phi}^4 + 2(1-\eta)\left[1-(2-\eta)||\mathbf{G}^T||_h^2\right]\bar{g}s\tilde{\phi}^3\\ &+\left\{\left[1-2(1-\eta)||\mathbf{G}^T||_h^2\right]-(2-\eta)^2\bar{g}^2s^2\right\}\tilde{\phi}^2 - 2(2-\eta)\bar{g}s\tilde{\phi} - 1 = 0.\end{aligned} \tag{7.27}$$

The last relation is derived from Eq. (7.26). Since the slippery slope metric $\tilde{F}_\eta$, for each $\eta \in [0,1]$, is a Finsler metric with

$$||\mathbf{G}^T||_h < \tilde{b}_0, \text{ where } \tilde{b}_0 = \begin{cases} 1, & \eta \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2\eta}, & \eta \in \left(\frac{1}{2}, 1\right] \end{cases}, \tag{7.28}$$

and applying Proposition 6.2.1, the function $\tilde{\phi}_\eta$ satisfies the following inequalities

$$\tilde{\phi}_\eta - s\tilde{\phi}_{\eta 2} > 0, \qquad \bar{g}^2(\tilde{\phi}_\eta - s\tilde{\phi}_{\eta 2}) + (||\mathbf{G}^T||_h^2 - \bar{g}^2s^2)\tilde{\phi}_{\eta 22} > 0,$$

when $n \geq 3$ or only

$$\bar{g}^2(\tilde{\phi}_\eta - s\tilde{\phi}_{\eta 2}) + (||\mathbf{G}^T||_h^2 - \bar{g}^2s^2)\tilde{\phi}_{\eta 22} > 0,$$

when $n = 2$, for any s such that $|s| \leq \frac{||\mathbf{G}^T||_h}{\bar{g}} < \frac{\tilde{b}_0}{\bar{g}}$. In order to arrive at the equations of the geodesics corresponding to the slippery slope metric $\tilde{F}_\eta$, for each $\eta \in [0,1]$, we need to determine the spray coefficients of $\tilde{F}_\eta$. A key ingredient for this is Proposition 6.2.2.

We now work toward the establishment of some relations between the function $\tilde{\phi}_{\tilde{\eta}}$ and its derivatives.

Lemma 7.2.4. *Let M be an n-dimensional manifold, $n > 1$, with the slippery slope metric $\tilde{F}_\eta(x,y) = \alpha\tilde{\phi}_\eta(||\mathbf{G}^T||_h^2, s)$. The function $\tilde{\phi}_\eta$ and its derivative with respect to s, i.e. $\tilde{\phi}_{\eta 2}$ hold the following relations*

$$C\tilde{\phi}_{\eta 2} = A\tilde{\phi}_\eta, \qquad C(\tilde{\phi}_\eta - s\tilde{\phi}_{\eta 2}) = B, \qquad C\tilde{\phi}_\eta = B + As\tilde{\phi}_\eta, \tag{7.29}$$

or each $\eta \in [0,1]$, *where*

$$\begin{aligned} A &= -2(1-\eta)[1-(2-\eta)||\mathbf{G}^T||_h^2]\bar{g}\tilde{\phi}_\eta^2 + 2(2-\eta)^2\bar{g}^2 s\tilde{\phi}_\eta + 2(2-\eta)\bar{g}, \\ B &= -2[1-2(1-\eta)||\mathbf{G}^T||_h^2]\tilde{\phi}_\eta^2 + 4(2-\eta)\bar{g}s\tilde{\phi}_\eta + 4, \\ C &= 4(1-\eta)^2||\mathbf{G}^T||_h^2(1-||\mathbf{G}^T||_h^2)\tilde{\phi}_\eta^3 + 6(1-\eta)[1-(2-\eta)||\mathbf{G}^T||_h^2]\bar{g}s\tilde{\phi}_\eta^2 \\ &\quad + 2\left\{[1-2(1-\eta)||\mathbf{G}^T||_h^2] - (2-\eta)^2\bar{g}^2 s^2\right\}\tilde{\phi}_\eta - 2(2-\eta)\bar{g}s, \end{aligned} \tag{7.30}$$

A, B, C being evaluated at $(||\mathbf{G}^T||_h^2, s)$. *Moreover,*
i) $C(||\mathbf{G}^T||_h^2, s) \neq 0$, $\tilde{\phi}_{\eta 2} = \frac{\bar{g}A}{C}\tilde{\phi}_\eta$, *and* $\tilde{\phi}_\eta - s\tilde{\phi}_{\eta 2} = \frac{B}{C}$.
ii) $B(||\mathbf{G}^T||_h^2, s) \neq 0$.

Proof. Since $\tilde{\phi}_\eta$ is a root of (7.27), it checks this identically, namely

$$\begin{aligned} &(1-\eta)^2||\mathbf{G}^T||_h^2(1-||\mathbf{G}^T||_h^2)\tilde{\phi}_\eta^4 + 2(1-\eta)\left[1-(2-\eta)||\mathbf{G}^T||_h^2\right]\bar{g}s\tilde{\phi}_\eta^3 \\ &+\left\{\left[1-2(1-\eta)||\mathbf{G}^T||_h^2\right] - (2-\eta)^2\bar{g}^2 s^2\right\}\tilde{\phi}_\eta^2 - 2(2-\eta)\bar{g}s\tilde{\phi}_\eta - 1 = 0. \end{aligned} \tag{7.31}$$

The derivative of the identity (7.31) with respect to s leads to the first relation from (7.29). Then, it immediately results the second identity of (7.29). The last one is justified by the notations (7.30) and (7.31).

Now in order to prove i) we suppose, towards a contradiction, that there exists $s_0 \in [-b,b]$, $b = \frac{||\mathbf{G}^T||_h}{\bar{g}} < \frac{\tilde{b}_0}{\bar{g}}$, with $\tilde{b}_0$ given by (7.28), such that $C(||\mathbf{G}^T||_h^2, s_0) = 0$. Under this assumption, the relations (7.29) imply $A(||\mathbf{G}^T||_h^2, s_0) = B(||\mathbf{G}^T||_h^2, s_0) = 0$. If we substitute this outcome in the identity (7.31), we obtain

$$(1-\eta)^2||\mathbf{G}^T||_h^2(1-||\mathbf{G}^T||_h^2)\tilde{\phi}_\eta^4 + [(2-\eta)\bar{g}s_0\tilde{\phi}_\eta + 1]^2 = 0, \tag{7.32}$$

which is impossible. Indeed, if $\eta \neq 1$, then $(1-\eta)^2||\mathbf{G}^T||_h^2(1-||\mathbf{G}^T||_h^2)\tilde{\phi}_\eta^4 \neq 0$ because $||\mathbf{G}^T||_h < 1$ and $\tilde{\phi}_\eta > 0$, or if $\eta = 1$, then $\bar{g}s_0\tilde{\phi}_\eta + 1 = \frac{1}{1-\bar{g}s_0} \neq 0$. Thus, $C \neq 0$ everywhere here.

To show the statement ii), we again argue by contradiction. Assume that there is $\tilde{s} \in [-b,b]$, $b = \frac{||\mathbf{G}^T||_h}{\bar{g}} < \frac{\tilde{b}_0}{\bar{g}}$, with $\tilde{b}_0$ given by (7.28), such that $B(||\mathbf{G}^T||_h^2, \tilde{s}) = 0$. Thus, we are searching for $\tilde{s}$ in the interval $[-b,b]$. If we take $s = \tilde{s}$ in the third formula in (7.29), an immediate consequence is $\tilde{s} \neq 0$, because of $\tilde{\phi}_\eta(||\mathbf{G}^T||_h^2, \tilde{s}) > 0$, $C(||\mathbf{G}^T||_h^2, \tilde{s}) \neq 0$ and $B(||\mathbf{G}^T||_h^2, \tilde{s}) = 0$. Moreover, under our assumption, the second formula in (7.30) turns out that $\tilde{\phi}_\eta(||\mathbf{G}^T||_h^2, \tilde{s})$ satisfies the polynomial equation

$$[1-2(1-\eta)||\mathbf{G}^T||_h^2]\tilde{\phi}_\eta^2 - 2(2-\eta)\bar{g}\tilde{s}\tilde{\phi}_\eta - 2 = 0 \tag{7.33}$$

and (7.31) is reduced to

$$\begin{aligned} &2(1-\eta)^2||\mathbf{G}^T||_h^2(1-||\mathbf{G}^T||_h^2)\tilde{\phi}_\eta^2 + \left[2-3\eta-2(2-\eta)(1-\eta)||\mathbf{G}^T||_h^2\right]\bar{g}\tilde{s}\tilde{\phi}_\eta \\ &+1-2(1-\eta)||\mathbf{G}^T||_h^2 = 0, \end{aligned} \tag{7.34}$$

for $s=\tilde{s}$ and for any $\eta\in[0,1]$. Since $||\mathbf{G}^T||_h<\tilde{b}_0$, with $\tilde{b}_0$ given by (7.28), $1-||\mathbf{G}^T||_h^2\neq 0$ for any $\eta\in[0,1]$, but may exist some $\eta\in[0,\frac{1}{2})$ such that $1-2(1-\eta)||\mathbf{G}^T||_h^2=0$. Thus, two cases must be distinguished.
a) if $1-2(1-\eta)||\mathbf{G}^T||_h^2\neq 0$, for any $\eta\in[0,1]$, then by (7.33) and (7.34) we obtain that $[1-4\eta(1-\eta)||\mathbf{G}^T||_h^2]\tilde{\phi}_\eta-4\eta\bar{g}\tilde{s}=0$, which provide a contradiction if $\eta=0$. Thus, $\eta\neq 0$ and taking into account that $\tilde{s}\neq 0$ and $\tilde{\phi}_\eta(||\mathbf{G}^T||_h^2,\tilde{s})>0$, we get $1-4\eta(1-\eta)||\mathbf{G}^T||_h^2\neq 0$ and

$$\tilde{\phi}_\eta(||\mathbf{G}^T||_h^2,\tilde{s})=\frac{4\eta\bar{g}\tilde{s}}{1-4\eta(1-\eta)||\mathbf{G}^T||_h^2}. \tag{7.35}$$

If we substitute (7.35) in (7.33) it yields $\tilde{s}^2=\frac{[1-4\eta(1-\eta)||\mathbf{G}^T||_h^2]^2}{4\eta\bar{g}^2\left[3\eta-2+4\eta(1-\eta)^2||\mathbf{G}^T||_h^2\right]}$ which contradicts $\tilde{s}^2\in(0,b^2]$, due to the condition $||\mathbf{G}^T||_h<\tilde{b}_0$, where $\tilde{b}_0$ is given by (7.28).
b) if $1-2(1-\eta)||\mathbf{G}^T||_h^2=0$ for some $\eta\in[0,\frac{1}{2})$, then (7.33) leads to

$$\tilde{\phi}_\eta(||\mathbf{G}^T||_h^2,\tilde{s})=-\frac{1}{(2-\eta)\bar{g}\tilde{s}}, \tag{7.36}$$

which together with (7.34) yields $-\eta\tilde{s}^2=\frac{1-2\eta}{4(2-\eta)\bar{g}^2}$. Obviously, the last relation provides a contradiction (i.e. $\tilde{s}^2<0$) for any $\eta\in[0,\frac{1}{2})$.

Summing up, we have $B(||\mathbf{G}^T||_h^2,s)\neq 0$, for any $s\in[-b,b]$, $b=\frac{||\mathbf{G}^T||_h}{\bar{g}}<\frac{\tilde{b}_0}{\bar{g}}$, with $\tilde{b}_0$ given by (7.28). □

We note that according to Proposition 6.2.1 we knew that $\tilde{\phi}_\eta-s\tilde{\phi}_{\eta 2}>0$, when $n\geq 3$, for any $\eta\in[0,1]$ and s such that $|s|\leq\frac{||\mathbf{G}^T||_h}{\bar{g}}<\frac{\tilde{b}_0}{\bar{g}}$. Now by Lemma 7.2.4, we have established that $\tilde{\phi}_\eta-s\tilde{\phi}_{\eta 2}\neq 0$ also when $n=2$, for any $\eta\in[0,1]$ and $|s|\leq\frac{||\mathbf{G}^T||_h}{\bar{g}}<\frac{\tilde{b}_0}{\bar{g}}$.

In addition, the functions A, B, C given in (5.11) are homogenous of degree zero with respect to y because of the same homogeneity degree of $\tilde{\phi}_\eta$, for any $\eta\in[0,1]$.

Lemma 7.2.5. *The derivatives of the function $\tilde{\phi}_\eta$ respect to $b^2=\frac{||\mathbf{G}^T||_h^2}{\bar{g}^2}$ and s, i.e. $\tilde{\phi}_{\eta 1}$, $\tilde{\phi}_{\eta 12}$ and $\tilde{\phi}_{\eta 22}$, respectively hold the following relations*

$$\begin{aligned}
&\tilde{\phi}_{\eta 1}=\frac{(1-\eta)\bar{g}^2}{2C}(B+2\eta\tilde{\phi}_\eta^2)\tilde{\phi}_\eta^2,\\
&\tilde{\phi}_{\eta 22}=\frac{1}{C^3}(A^2B+2D^2),\\
&\tilde{\phi}_{\eta 12}=\frac{(1-\eta)\bar{g}^2}{2C^3}\left[(B+2\eta\tilde{\phi}_\eta^2)(2AB+2DH+A^2s\tilde{\phi}_\eta)-4CD\tilde{\phi}_\eta\right]\tilde{\phi}_\eta,
\end{aligned} \tag{7.37}$$

and

$$\begin{aligned}
&s\tilde{\phi}_\eta+(b^2-s^2)\tilde{\phi}_{\eta 2}=\frac{1}{\bar{g}^2C}(B\bar{g}^2s+A||\mathbf{G}^T||_h^2\tilde{\phi}_\eta),\\
&(\tilde{\phi}_\eta-s\tilde{\phi}_{\eta 2})\tilde{\phi}_{\eta 2}-s\tilde{\phi}_\eta\tilde{\phi}_{\eta 22}=\frac{1}{C^3}(AB^2-2D^2s\tilde{\phi}_\eta),\\
&\tilde{\phi}_\eta-s\tilde{\phi}_{\eta 2}+(b^2-s^2)\tilde{\phi}_{\eta 22}=\frac{1}{\bar{g}^2C^3}[\bar{g}^2BC^2+(||\mathbf{G}^T||_h^2-\bar{g}^2s^2)(A^2B+2D^2)],\\
&(\tilde{\phi}_\eta-s\tilde{\phi}_{\eta 2})\tilde{\phi}_{\eta 12}-s\tilde{\phi}_{\eta 1}\tilde{\phi}_{\eta 22}=\frac{(1-\eta)\bar{g}^2}{C^4}[4\eta CD\tilde{\phi}_\eta^3+(B+2\eta\tilde{\phi}_\eta^2)(AB^2-2D^2s\tilde{\phi}_\eta)]\tilde{\phi}_\eta,
\end{aligned} \tag{7.38}$$

where $D=2A-(2-\eta)\bar{g}B$, $H=2+(2-\eta)\bar{g}s\tilde{\phi}_\eta$, for any $\eta\in[0,1]$.

Proof. By derivation with respect to $||\mathbf{G}^T||_h^2$, the identity (7.31) gives

$$\frac{\partial\tilde{\phi}_\eta}{\partial||\mathbf{G}^T||_h^2} = \frac{(1-\eta)}{2C}(B+2\eta\tilde{\phi}_\eta^2)\tilde{\phi}_\eta^2.$$

When substituted in $\tilde{\phi}_{\eta 1} = \bar{g}^2\frac{\partial\tilde{\phi}_\eta}{\partial||\mathbf{G}^T||_h^2}$, this yields the first expression in (7.37). The derivatives of (7.30) with respect to s read

$$A_2 = \frac{2}{C}[A^2-(2-\eta)\bar{g}D], \qquad B_2 = \frac{2}{C}[AB-2D], \qquad C_2 = -\frac{AB+2DH}{C\tilde{\phi}_\eta}+3A,$$

where $A_2 = \frac{\partial A}{\partial s}$, $B_2 = \frac{\partial B}{\partial s}$, $C_2 = \frac{\partial B}{\partial s}$. These, along with

$$\begin{aligned}\tilde{\phi}_{\eta 22} &= \frac{A_2C+A^2-AC_2}{C^2}\tilde{\phi}_\eta,\\ \tilde{\phi}_{\eta 12} &= \frac{(1-\eta)\bar{g}^2}{2C^2}\left[B_2C+4\eta\tilde{\phi}_\eta^2A+(B+2\eta\tilde{\phi}_\eta^2)(2A-C_2)\right]\tilde{\phi}_\eta^2,\end{aligned}$$

lead to the last two formulas in (7.37). Once we obtain $\tilde{\phi}_{\eta 1}$, $\tilde{\phi}_{\eta 2}$, $\tilde{\phi}_{\eta 12}$ and $\tilde{\phi}_{\eta 22}$, a technical computation yields the expressions (7.38). □

Beyond the force of the gravitational wind $\mathbf{G}^T = -\bar{g}\omega^\sharp$, we can emphasize some specific features of $\mathbf{G}^T$ which come from the properties of the gradient vector field $\omega^\sharp = h^{ji}\frac{\partial p}{\partial x^j}\frac{\partial}{\partial x^i}$; cf. [20]. Thus, denoting the components of $\mathbf{G}^T$ by w^i, these have the expressions $w^i = -\bar{g}h^{ji}\frac{\partial p}{\partial x^j}$. With the notation $w_i = h_{ij}w^j$, it follows that $w_i = -\bar{g}\frac{\partial p}{\partial x^i}$, having the property $\frac{\partial w_i}{\partial x^j} = \frac{\partial w_j}{\partial x^i}$.

Lemma 7.2.6. *For the gravitational wind* $\mathbf{G}^T$ *the following relations hold*

$$\begin{aligned}&r_{ij} = -\tfrac{1}{\bar{g}}w_{i|j}, \qquad r_i = \tfrac{1}{\bar{g}^2}w_{i|j}w^j, \qquad r^i = \tfrac{1}{\bar{g}^2}w^i{}_{|j}w^j, \qquad r = -\tfrac{1}{\bar{g}^3}w_{i|j}w^iw^j,\\ &r_{00} = -\tfrac{1}{\bar{g}}w_{i|j}y^iy^j, \qquad r_0 = \tfrac{1}{\bar{g}^2}w_{i|j}w^jy^i, \qquad s_{ij} = s_i = s^i = s^i_0 = s_0 = 0,\end{aligned} \tag{7.39}$$

where $w_{i|j} = \frac{\partial w_i}{\partial x^j} - \Gamma^k_{ij}w_k$, $w^i{}_{|j} = h^{ik}w_{k|j}$ *and* $\Gamma^k_{ij} = \frac{1}{2}h^{km}\left(\frac{\partial h_{jm}}{\partial x^i}+\frac{\partial h_{im}}{\partial x^j}-\frac{\partial h_{ij}}{\partial x^m}\right)$. *In addition,* $||\mathbf{G}^T||_h$ *is constant if and only if* $r_i = 0$ *and, under either of the statements of this equivalence,* $r^i = r = r_0 = 0$.

Proof. Taking into account (7.14), it implies that

$$a_{ij} = h_{ij}, \;\; b_i = -\frac{1}{\bar{g}}w_i = \frac{\partial p}{\partial x^i}, \;\; b^i = h^{ji}b_j = -\frac{1}{\bar{g}}w^i$$

and moreover, $\frac{\partial b_i}{\partial x^j} = \frac{\partial b_j}{\partial x^i}$, $b_{i|j} = b_{j|i}$ (i.e. β is closed) and $b_{i|j} = -\frac{1}{\bar{g}}w_{i|j}$.

Making use of (6.4), the conditions in (7.39) are fulfilled. A trivial computation shows that $\frac{\partial||\mathbf{G}^T||_h}{\partial x^i} = \frac{2}{\bar{g}^2}w_{i|j}w^j = 2r_i$. This clearly forces $r_i = 0$ if and only if $||\mathbf{G}^T||_h$ is constant. □

Proposition 7.2.7. *Let M be an n-dimensional manifold, $n>1$, with the slippery slope metric $\tilde{F}_\eta$, with $\eta\in[0,1]$. The spray coefficients $\tilde{\mathcal{G}}^i_\eta$ of $\tilde{F}_\eta$ are related to the spray coefficients $\mathcal{G}^i_\alpha=\frac{1}{4}h^{im}\left(2\frac{\partial h_{jm}}{\partial x^k}-\frac{\partial h_{jk}}{\partial x^m}\right)y^jy^k$ of α by*

$$\tilde{\mathcal{G}}^i_\eta(x,y)=\mathcal{G}^i_\alpha(x,y)+\left[\Theta(r_{00}+2\alpha^2Rr)+\alpha\Omega r_0\right]\frac{y^i}{\alpha}-\left[\Psi(r_{00}+2\alpha^2Rr)+\alpha\Pi r_0\right]\frac{w^i}{\bar{g}}-\alpha^2Rr^i, \tag{7.40}$$

where

$$R=\frac{(1-\eta)\bar{g}^2}{2B\alpha^4}(B\alpha^2+2\eta\tilde{F}^2_\eta)\tilde{F}^2_\eta,\quad \Theta=\frac{\bar{g}^2\alpha(AB^2\alpha^2-2D^2\beta\tilde{F}_\eta)}{2E\tilde{F}_\eta},\quad \Psi=\frac{\bar{g}^2\alpha^2(A^2B+2D^2)}{2E},$$

$$\Omega=\frac{(1-\eta)\bar{g}^2}{BE\alpha^2}[(B\alpha^2+2\eta\tilde{F}^2_\eta)(\bar{g}^2B^3\alpha^2+2D^2||\mathbf{G}^T||^2_h\tilde{F}^2_\eta)-4\eta D\tilde{F}^3_\eta(\bar{g}^2B\beta+A||\mathbf{G}^T||^2_h\tilde{F}_\eta)],$$

$$\Pi=\frac{(1-\eta)\bar{g}^4}{BE\alpha^3}[4\eta CD\alpha\tilde{F}^3_\eta+(B\alpha^2+2\eta\tilde{F}^2_\eta)(AB^2\alpha^2-2D^2\beta\tilde{F}_\eta)]\tilde{F}_\eta, \tag{7.41}$$

with

$$\begin{aligned}
A&=-\frac{2\bar{g}}{\alpha^2}\left\{(1-\eta)[1-(2-\eta)||\mathbf{G}^T||^2_h]\tilde{F}^2_\eta-(2-\eta)^2\bar{g}\beta\tilde{F}_\eta-(2-\eta)\alpha^2\right\},\\
B&=-\frac{2}{\alpha^2}\left\{[1-2(1-\eta)||\mathbf{G}^T||^2_h]\tilde{F}^2_\eta-2(2-\eta)\bar{g}\beta\tilde{F}_\eta-2\alpha^2\right\},\\
C&=\frac{1}{\alpha\tilde{F}_\eta}\left(B\alpha^2+A\beta\tilde{F}_\eta\right),\quad D=2A-(2-\eta)\bar{g}B,\\
E&=\bar{g}^2BC^2\alpha^2+(||\mathbf{G}^T||^2_h\alpha^2-\bar{g}^2\beta^2)(A^2B+2D^2)
\end{aligned} \tag{7.42}$$

and the formulae for r_{00}, r_0, r and r^i are given in (7.39).

Proof. The proof follows from Proposition 6.2.2 and Lemmas 7.2.5 and 7.2.6. □

Note that if $||\mathbf{G}^T||^2_h$ is constant, then the formula (7.40) is simplified considerably, i.e.

$$\tilde{\mathcal{G}}^i_\eta(x,y)=\mathcal{G}^i_\alpha(x,y)+r_{00}\left(\Theta\frac{y^i}{\alpha}-\Psi\frac{w^i}{\bar{g}}\right), \tag{7.43}$$

because $r^i=r=r_0=0$ in this particular case.

Therefore, owing to the system (6.2) and Proposition 7.2.7 with $\tilde{F}_\eta(\gamma(t),\dot{\gamma}(t))=1$, we can write the ODE system (7.5) which yields the shortest time trajectories $\gamma(t)=(\gamma^i(t))$, $i=1,...,n$ on the slippery slope of a mountain. This ends the proof of Theorem 7.1.2.

Finally, we apply the general theory developed in this chapter by emphasizing a two-dimensional example, namely Gaussian bell-shaped surface. First, we give a brief overview of the general model of the hill slope in dimension 2 which is described in [10, 11, 12, 13]. Coming back to the particular case with M being a surface, where π_O is the tangent plane to M at $O\in M$, the parametric equations of the indicatrix of the slippery slope metric $\tilde{F}_\eta$ in the coordinates (X,Y) with respect to the rectangular basis $\{e_1,e_2\}$ are given by

$$\begin{cases} X = (1+\eta||\mathbf{G}^T||_h\cos\theta)\cos\theta+(1-\eta)||\mathbf{G}^T||_h \\ Y = (1+\eta||\mathbf{G}^T||_h\cos\theta)\sin\theta \end{cases}, \tag{7.44}$$

for any traction coefficient $\eta \in [0,1]$ and the direction of the self-velocity $\theta \in [0,2\pi)$, because $\mathbf{G}^T$ and e_1 point in the same direction. If we eliminate θ in (7.44), it follows the equation of the indicatrix of $\tilde{F}_\eta$, namely

$$\sqrt{(X-(1-\eta)||\mathbf{G}^T||_h)^2+Y^2} = X^2+Y^2-(2-\eta)X||\mathbf{G}^T||_h+(1-\eta)||\mathbf{G}^T||_h^2. \qquad (7.45)$$

Considering the surface M embedded in $\mathbb{R}^3$ and parametrized by $(x^1,x^2)\in M \to (x=x^1, y=x^2, z=f(x^1,x^2))\in\mathbb{R}^3$, where f is a smooth function on M, the Riemannian metric induced on M is $\left(h_{ij}(x^1,x^2)\right)=\begin{pmatrix} 1+f_{x^1}^2 & f_{x^1}f_{x^2} \\ f_{x^1}f_{x^2} & 1+f_{x^2}^2\end{pmatrix}$, $i,j=1,2$. The notations f_{x^1} and f_{x^2} mean the partial derivatives of f with respect to x^1 and x^2, respectively. Thus, the tangent plane π_O to M is spanned by the vectors $\frac{\partial}{\partial x^1}=(1,0,f_{x^1})$ and $\frac{\partial}{\partial x^2}=(0,1,f_{x^2})$, and the gravitational wind is

$$\mathbf{G}^T=-\frac{\bar{g}}{q+1}(f_{x^1},f_{x^2},q)=-\frac{\bar{g}}{q+1}\left(f_{x^1}\frac{\partial}{\partial x^1}+f_{x^2}\frac{\partial}{\partial x^2}\right), \qquad (7.46)$$

with $||\mathbf{G}^T||_h=\bar{g}\sqrt{\frac{q}{q+1}}$, where $q=f_{x^1}^2+f_{x^2}^2$. Since any tangent vector of π_O can be written as $y^1\frac{\partial}{\partial x^1}+y^2\frac{\partial}{\partial x^2}=Xe_1+Ye_2$, with $e_1=-\frac{1}{\sqrt{q(q+1)}}(f_{x^1},f_{x^2},q)$ and $e_2=\frac{1}{\sqrt{q}}(f_{x^2},-f_{x^1},0)$. Thus, it turns out the following link between the coordinates (X,Y) and (y^1,y^2)

$$X=-\sqrt{\frac{q+1}{q}}\left(y^1f_{x^1}+y^2f_{x^2}\right), \qquad Y=\frac{1}{\sqrt{q}}\left(y^1f_{x^2}-y^2f_{x^1}\right). \qquad (7.47)$$

Furthermore, since $y^1f_{x^1}+y^2f_{x^2}=-\frac{1}{\bar{g}}h(y,\mathbf{G}^T)$, we obtain

$$X^2+Y^2=(y^1)^2+(y^2)^2+\beta^2=h_{ij}y^iy^j=\alpha^2, \quad y^1f_{x^1}+y^2f_{x^2}=\beta. \qquad (7.48)$$

When substituted in (7.45) this yields

$$\sqrt{\alpha^2+2(1-\eta)\bar{g}\beta+(1-\eta)^2||\mathbf{G}^T||_h^2}=\alpha^2+(2-\eta)\bar{g}\beta+(1-\eta)||\mathbf{G}^T||_h^2. \qquad (7.49)$$

As a consequence, by Okubo's method we arrive at the equation (7.24) that gives the slippery slope metric $\tilde{F}_\eta$. If O is a critical point of M, i.e. a point where $q(O)=0$, then $\mathbf{G}^T=0$. Although the above slippery slope metric $\tilde{F}_\eta$ is described only at regular points O of the surface M ($q(O)\neq 0$), it is well defined everywhere on M including the critical points of M, where it is just the background Riemannian metric h.

Let $\mathfrak{G}$ be a surface of revolution described by the two-dimensional Gaussian function $z=\frac{3}{2}e^{-(x^2+y^2)}$ (i.e. Gaussian bell-shaped surface). Corresponding to $\mathfrak{G}$ the gravitational wind (7.46) is

$$\mathbf{G}^T=\frac{3\bar{g}e^{-(x^2+y^2)}}{9(x^2+y^2)e^{-2(x^2+y^2)}+1}\left(x,y,-3(x^2+y^2)e^{-(x^2+y^2)}\right),$$

because $f(x^1,x^2)=\frac{3}{2}e^{-(x^2+y^2)}$, where $x=x^1$, $y=x^2$ and $q=9(x^2+y^2)e^{-2(x^2+y^2)}$ [10, 11, 12]. For simplicity, we choose the following parametrization for the surface of revolution

$$\mathfrak{G}:(\rho,\varphi)\in\mathfrak{G}\to(x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ z=\frac{3}{2}e^{-\rho^2})\in\mathbb{R}^3,$$

where $\rho \geq 0$ and $\varphi \in [0, 2\pi)$. This leads to

$$\mathbf{G}^T(\rho,\varphi) = \frac{3\bar{g}\rho e^{-\rho^2}}{9\rho^2 e^{-2\rho^2}+1}\frac{\partial}{\partial\rho}, \quad \text{where} \quad ||\mathbf{G}^T||_h = \frac{3\bar{g}\rho e^{-\rho^2}}{\sqrt{9\rho^2 e^{-2\rho^2}+1}}. \tag{7.50}$$

Since $h_{11}(\rho,\varphi) = 9\rho^2 e^{-2\rho^2}+1$, $h_{22}(\rho,\varphi) = \rho^2$ and $h_{12}(\rho,\varphi) = h_{21}(\rho,\varphi) = 0$, some technical computation yields

$$\alpha^2 = (9\rho^2 e^{-2\rho^2}+1)\dot{\rho}^2 + \rho^2\dot{\varphi}^2, \qquad \beta = -3\rho e^{-\rho^2}\dot{\rho}, \tag{7.51}$$

$$\mathcal{G}^1_\alpha = \frac{\rho}{2(9\rho^2 e^{-2\rho^2}+1)}\left[9(1-2\rho^2)e^{-2\rho^2}\dot{\rho}^2 - \dot{\varphi}^2\right], \qquad \mathcal{G}^2_\alpha = \frac{1}{\rho}\dot{\rho}\dot{\varphi} \tag{7.52}$$

$$\begin{aligned} r_{00} &= -\tfrac{3e^{-\rho^2}}{9\rho^2 e^{-2\rho^2}+1}\left[(1-2\rho^2)\dot{\rho}^2+\rho^2\dot{\varphi}^2\right], \qquad r_0 = \tfrac{9\rho(1-2\rho^2)e^{-2\rho^2}}{(9\rho^2 e^{-2\rho^2}+1)^2}\dot{\rho} \\ r &= -\tfrac{27\rho^2(1-2\rho^2)e^{-3\rho^2}}{(9\rho^2 e^{-2\rho^2}+1)^3}, \qquad r^1 = \tfrac{9\rho(1-2\rho^2)e^{-2\rho^2}}{(9\rho^2 e^{-2\rho^2}+1)^3}, \qquad r^2 = 0. \end{aligned} \tag{7.53}$$

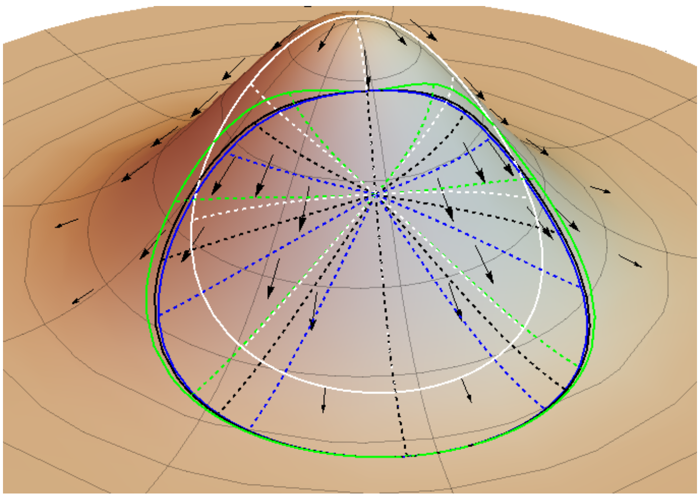


Figure 7.2: The slippery slope geodesics on $\mathfrak{G}$ with the boundary cases: $\eta = 0.5$ (dashed black), $\eta = 1$ (Matsumoto, dashed green) and $\eta = 0$ (Zermelo-Randers, dashed blue). They are compared to the Riemannian geodesics (dashed white). The corresponding unit time fronts are shown in solid colours and the gravitational wind $\mathbf{G}^T$ (black arrows) "blows" in the steepest downhill direction; $\bar{g} = 0.63$, $\Delta t = 1$, with a step of $\Delta\theta = \pi/4$ (8 paths in each case) and the initial point is positioned on the parallel of the strongest gravitational wind, i.e. $(\rho(0), \varphi(0)) = (1/\sqrt{2}, -\pi/4)$.

Lemma 7.2.8. *The indicatrix of the slippery slope metric $\tilde{F}_\eta$ on the surface $\mathfrak{G}$ is strongly convex if and only if $\bar{g} < \delta_2(\eta)$, where* $\delta_2(\eta) = \begin{cases} \frac{\sqrt{2e+9}}{3} \approx 1.27, & \text{if } \eta \in [0, \frac{1}{2}] \\ \frac{\sqrt{2e+9}}{6\eta} \approx \frac{0.64}{\eta}, & \text{if } \eta \in (\frac{1}{2}, 1] \end{cases}$.

Proof. Since $||\mathbf{G}^T||_h = \frac{3\bar{g}\rho e^{-\rho^2}}{\sqrt{9\rho^2 e^{-2\rho^2}+1}}$, its maximum value is $\frac{3\bar{g}}{\sqrt{2e+9}} \approx 0.79\bar{g}$, for any $\rho \geq 0$, and it is achieved when $\rho = \frac{1}{\sqrt{2}} \approx 0.71$. Hence, $||\mathbf{G}^T||_h \leq \frac{3\bar{g}}{\sqrt{2e+9}}$, for any $\rho \geq 0$. Thus, $\bar{g} < \delta_2(\eta)$ is equivalent to the strong convexity condition (7.28). □

It is easily seen that the most restrictive case regarding the convexity among all slippery slope metrics on $\mathfrak{G}$ is for $\eta = 1$, that is, the solution of the standard Matsumoto problem. Next, owing to Theorem 7.1.2 and Proposition 7.2.7, the time geodesics $\gamma(t) = (\rho(t), \varphi(t))$ on the slippery slope $\mathfrak{G}$ are the solutions of the ODE system

$$\begin{cases} 0 = \ddot{\rho} + \frac{\rho}{9\rho^2 e^{-2\rho^2}+1}\left[9(1-2\rho^2)e^{-2\rho^2}\dot{\rho}^2 - \dot{\varphi}^2\right] + 2\left\{\tilde{\Theta}(r_{00} + 2\alpha^2\tilde{R}r) + \alpha\tilde{\Omega}r_0\right\}\frac{\dot{\rho}}{\alpha} \\ \quad -\frac{6\rho e^{-\rho^2}}{9\rho^2 e^{-2\rho^2}+1}\left\{\tilde{\Psi}(r_{00} + 2\alpha^2\tilde{R}r) + \alpha\tilde{\Pi}r_0\right\} - \frac{18\rho(1-2\rho^2)e^{-2\rho^2}}{(9\rho^2 e^{-2\rho^2}+1)^3}\alpha^2\tilde{R} \\ 0 = \ddot{\varphi} + \frac{2}{\rho}\dot{\rho}\dot{\varphi} + 2\left\{\tilde{\Theta}(r_{00} + 2\alpha^2\tilde{R}r) + \alpha\tilde{\Omega}r_0\right\}\frac{\dot{\varphi}}{\alpha} \end{cases},$$

where $\tilde{\Theta}$, $\tilde{R}$, $\tilde{\Omega}$, $\tilde{\Pi}$ and $\tilde{\Psi}$ are given by (7.6), $||\mathbf{G}^T||_h = \frac{3\bar{g}\rho e^{-\rho^2}}{\sqrt{9\rho^2 e^{-2\rho^2}+1}}$, $\bar{g} < \delta_2(\eta)$, $\rho = \rho(t)$, $\varphi = \varphi(t)$, with (7.51) and (7.53).

The outcome is presented in Figure 7.2, where the slippery slope geodesics ($\eta = 0.5$, dashed black) are compared with the boundary cases, i.e. the standard Matsumoto geodesics ($\eta = 1$, dashed green) and the Zermelo-Randers ($\eta = 0$, dashed blue) under the gravitational wind $\mathbf{G}^T$; $\bar{g} = 0.63 < \delta_2(\eta = 1)$. Moreover, the corresponding unit time fronts are shown in the respective solid colours. As expected, the time geodesics and fronts referring to $\eta = 1/2$ lie entirely between the corresponding Matsumoto and Randers paths.

Chapter 8

The slope-of-a-mountain problem in a cross gravitational wind

In this chapter, mainly based on the papers [11, 12], we describe additional models of slope as a Riemannian manifold, continuing to explore the influences of both transverse and longitudinal components of the gravitational wind on time-optimal paths. In contrast to the original Matsumoto's exposition, in the first model presented below, the cross-gravity additive is taken into account in the equations of motion, while the along-gravity effect is entirely compensated. Exploring the properties of the cross slope metric obtained in this work, which belongs to the class of general (α, β)-metrics and serves as a main tool, we find the time geodesics on a mountain slope under the influence of a cross gravitational wind. The feature of the second model, which generalizes and includes the first, is that the varying along-gravity effect depends on traction, whereas the cross-gravity additive is taken entirely in the equations of motion, for any direction and gravity force. The investigation of this also leads to a general (α, β)-metric called slippery-cross-slope metric which enables us to provide the corresponding time geodesics as well as to create a direct link between the Zermelo navigation problem and the slope-of-a-mountain problem under the action of a cross gravitational wind.

8.1 Model of a slope under the cross-gravity effect

Recalling the original Matsumoto's reasoning [106], one can observe that the slope-of-a-mountain problem was actually studied only under the influence of the longitudinal component of a gravitational wind $\mathbf{G}^T$ [1], which is collinear with self-velocity vector u (the control vector) of a walker, whereas the effect of another (transverse[2]) component was not taken into consideration in the model. Namely, the latter was assumed to be always cancelled and it did not have therefore any influence on the trajectory, although the orthogonal projection of $\mathbf{G}^T$ on $u^\perp$, denoted by $\mathrm{Proj}_{u^\perp}\mathbf{G}^T$ is in general a nonzero vector, because $\mathbf{G}^T = \mathrm{Proj}_u\mathbf{G}^T + \mathrm{Proj}_{u^\perp}\mathbf{G}^T$. More precisely, this issue was justified by Matsumoto in a word, saying that "*the component perpendicular to the velocity u is regarded to be cancelled by planting the walker's legs on the road determined by u*" [106, p. 19]. Consequently, the resulting velocity then reads $v = u + \mathrm{Proj}_u\mathbf{G}^T$, with $||v||_h = ||u||_h \pm ||\mathrm{Proj}_u\mathbf{G}^T||_h$ (+/- for a downhill/uphill path, respec-

[1] In short, a *gravitational wind* is the component $\mathbf{G}^T$ of a gravitational field $\mathbf{G} = \mathbf{G}^T + \mathbf{G}^\perp$, which is tangent to a slope and acts along the steepest downhill direction (see Figure 8.1).

[2] That is, collinear with $u^\perp$, which is a perpendicular direction to u.

tively). However, note that the impact[3] of the lateral component on the resulting velocity can be stronger than the longitudinal one, depending on the desired direction of motion on the hillside. This motivated us to consider and to compare a different scenario on the slope including the cross-gravity effect, and not the along-gravity additive like in [106]. The geometric construction of the corresponding Finslerian indicatrix in the new setting will also be based on a direction-dependent deformation of the background Riemannian metric. However, the general equations of motion will read now

$$v = u + \text{Proj}_{u^\perp}\mathbf{G}^T.$$

As a consequence, the velocities u and v are not collinear in general, which is in contrast to Matsumoto's model[4]. Such set-up refers in reality to a walker on a slope, who endeavors to keep the effective speed constant[5]. Namely, $||\text{Proj}_u v||_h$ is equal to the self-speed $||u||_h$ continuously on the slope by compensating the influence of the along-gravity additive $\text{Proj}_u\mathbf{G}^T$ that pushes the walker forward (when going downhill) or backward (when going uphill), and at the same time allowing the walker to be dragged off the direction pointed by u to the side by gravity. There is a close analogy with the linear transverse vessel's sliding motion side-to-side called sway on a dynamic surface of a sea (generated by wind, water waves or the inertia of a ship), while the linear front-back motion called surge is stabilized (compensated). Moreover, this can also be compared to the craft's (e.g. a vessel, an airplane) lateral drift from its course. In nature such type of motion has some analogy with the animals' behaviour that can move sideways, while being influenced by a natural force field, e.g. a sidewinder rattlesnake, or a hummingbird. Furthermore, this new setting gives rise to the description of some Finslerian indicatrices as the algebraic (pedal) curves and surfaces; see for instance [64] in this regard.

8.1.1 Cross gravitational wind

Before stating the title problem and formulating the main results we begin by briefly recalling some basic concepts, which have been introduced in [10, 11, 12] and setting up terminology and notation.

In this subsection, (M, h) is a surface embedded in $\mathbb{R}^3$, i.e. a 2-dimensional Riemannian manifold. Let π_O be the tangent plane to M at an arbitrary point $O \in M$. Considering that $\mathbf{G}$ is a gravitational field in $\mathbb{R}^3$ that affects a mountain slope M, this can be decomposed into two orthogonal components, $\mathbf{G} = \mathbf{G}^T + \mathbf{G}^\perp$, where $\mathbf{G}^\perp$ is normal and $\mathbf{G}^T$ tangent to M in O. The gravitational wind $\mathbf{G}^T$ acts along an anti-gradient, i.e. the steepest descent direction and its norm with respect to h is $||\mathbf{G}^T||_h = \sqrt{h(\mathbf{G}^T, \mathbf{G}^T)}$; see Figure 8.1. In general, $\mathbf{G}^T$ depends on the gradient vector field related to slope M and a given acceleration of gravity. Furthermore, we can decompose the gravitational wind as $\mathbf{G}^T = \overrightarrow{OA} + \overrightarrow{OA'}$, where $\overrightarrow{OA}$ is the orthogonal projection of $\mathbf{G}^T$ on the self-velocity u, denoted by $\mathbf{G}_{MAT}$, and its active, i.e. non-compensated part is an effective wind. Moreover, the second component $\overrightarrow{OA'}$ represents a

[3]That is, "force" of the gravitational wind component expressed by its norm w.r.t. the background Riemannian metric h.

[4]Both velocities are collinear only if the steepest route is followed, i.e. the gradient (uphill) or anti-gradient (downhill) direction. Moreover, $v = u$, since the cross gravitational wind $\mathbf{G}_\dagger = \overrightarrow{0}$ in these particular cases.

[5]The effective (longitudinal) speed of a walker, i.e. $||u||_h \pm ||Proj_u\mathbf{G}^T||_h$ is the same now like the resultant speed $||v||_h$ when walking on a horizontal plane, where the gravity acts perpendicularly on this plane, i.e. $\mathbf{G}^\perp = \mathbf{G}$. So, $\mathbf{G}^T = \overrightarrow{0}$ and thus, for any direction, $||v||_h = ||u||_h$ in this case. In general, we have $||v||_h = \sqrt{1 + ||\text{Proj}_{u^\perp}\mathbf{G}^T||_h^2}$.

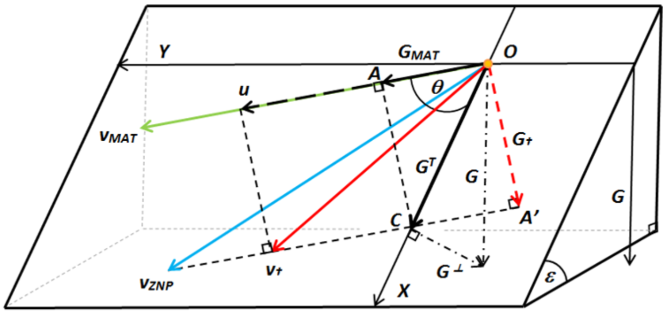


Figure 8.1: A model of a planar slope (an inclined plane of the slope angle ε) under gravity $\mathbf{G} = \mathbf{G}^T + \mathbf{G}^\perp$, which acts perpendicularly on the base of the slope (the horizontal plane), where $\mathbf{G}^T$ is the gravitational wind "blowing" tangentially to the slope in the steepest downhill direction X and $\mathbf{G}^\perp$ is the component of gravity normal to the slope; $OX \perp OY$. The resultant velocity $v_\dagger$ (red), which determines the motion on the slope under the cross wind $\mathbf{G}_\dagger$ (dashed red), is compared to the tangent vectors to the Zermelo-Randers (v_{ZNP}, blue) and Matsumoto (v_{MAT}, green) geodesics. The walker's self-velocity u (in other words, the Riemannian velocity) is marked in dashed black, $||u||_h = 1$. The impact of either orthogonal component of gravitational wind can be stronger than the other one, depending on the direction of motion θ on the slope; $OA \perp OA'$. In the original Matsumoto's setting the equation of motion reads $v = u + \mathbf{G}_{MAT}$, where the cross-gravity effect of $\mathbf{G}_\dagger$ is cancelled. Now, in contrast to [106] this yields $v = u + \mathbf{G}_\dagger$, where $||\mathrm{Proj}_u v||_h = ||u||_h$, since the along-gravity effect of $\mathbf{G}_{MAT}$ is compensated entirely.

cross gravitational wind, denoted here by $\mathbf{G}_\dagger$[6]. Their norms generally depend on the direction of motion θ and gravitational wind force $||\mathbf{G}^T||_h$. Thus, it is obvious that $||\mathbf{G}_\dagger||_h \in \left[0, ||\mathbf{G}^T||_h\right]$.

In contrast to Matsumoto's exposition, we do not assume that while the Earth's gravity acts on a walker on the slope, the cross wind perpendicular to a desired direction of motion (represented by a control vector u) is regarded to be cancelled. However, the effective wind, which pushes a walker downhill, is compensated completely regardless of the direction of motion, so its actual outcome is reduced to 0. In this new setting the sideways effect caused by gravity is taken into consideration instead. Thus, the influences of both components of the gravitational wind are reversed in comparison to [106]. In other words, the proposed model refers to a slope of a hill or a mountain, admitting the entire cross-track additive while compensating the along-track changes at the same time.

Moreover, observe that in the new set-up the resultant speed on the slope is always greater or equal to the self-speed of the walker, for any direction of motion θ. This property also differs from the situation in the standard Matsumoto problem as well as the Zermelo navigation problem on the slope, where the resultant speeds can be both higher or lower than unit own speed, depending on the direction. Only in the special cases, i.e. $\theta \in \{0, \pi\}$ (the steepest downhill/uphill paths) it is equal to the self-speed $||u||_h$, so like walking on a flat area, where the gravitational wind vanishes (the Riemannian case). Furthermore, after having paid a little thought, if $\theta \in \{\pi/2, 3\pi/2\}$, then we can see that such special case [7] coincides with

[6]To be precise, $\overrightarrow{OA}$ is in general the maximum effective wind and $\overrightarrow{OA'}$ the maximum cross wind, for given θ and $||\mathbf{G}^T||_h$. A component of the gravitational wind $\mathbf{G}^T$ is maximal if it is not compensated (reduced) partially or entirely, e.g. due to traction, drag.

[7]Such orientation of u resembles traversing a mountain slope along an isohypse, where $||v||_h = ||u||_h$ in Matsumoto's model in this case.

the navigation problem of Zermelo in the presence of a weak wind $\mathbf{G}^T$. This also yields the maximum possible speed $||v||_h$ (for a given $\mathbf{G}^T$) in the problem presented because $\mathbf{G}_\dagger = \mathbf{G}^T$ in this case. Therefore[8], $||v||_h \in [||u||_h, \sqrt{||u||_h^2 + ||\mathbf{G}^T||_h^2}]$.

It follows clearly that the cross gravitational wind can be expressed here as

$$\mathbf{G}_\dagger = -\mathbf{G}_{MAT} + \mathbf{G}^T. \tag{8.1}$$

Because of the cross-gravity effect, the self-velocity u is perturbed by $\mathbf{G}_\dagger$. Hence, the resultant velocity will be given by the composed vector

$$v = u + \mathbf{G}_\dagger. \tag{8.2}$$

This general relation defines the equation of motion on the slope under the influence of cross wind.

For comparison, it is worth pointing out that if the longitudinal component of $\mathbf{G}^T$ was not compensated at all in the model, then such setting would in general yield a scenario like in Zermelo's problem under a weak gravitational wind, i.e. with the navigation data $W = \mathbf{G}^T$, where the solution is given by Finsler metric of Randers type [127, 45, 10, 157]. This case demonstrates the action of the entire gravitational wind, so the full effects of its both components are then admitted on the slope.

8.1.2 The main results

Bearing in mind the ones above stated, the slope-of-a-mountain problem under the cross-gravity effect can be formulated as follows:

> *Suppose a person walks on a horizontal plane at a constant speed, while gravity acts perpendicularly on this plane. Imagine the person walks now on a slope of a mountain under the influence of a cross gravitational wind. How should the person navigate on the slope to get from one point to another in the shortest time?*

As the answer in the general context of an n-dimensional Riemannian manifold with $\mathbf{G}^T = -\bar{g}\omega^\sharp$ below we formulate our main theorem, where $\omega^\sharp$ is the gradient vector field and $\bar{g}$ is the rescaled gravitational acceleration g.

Theorem 8.1.1. (Cross-slope metric) *Let the slope of a mountain be an n-dimensional Riemannian manifold (M, h), $n > 1$, with the gravitational wind $\mathbf{G}^T$. The time-minimal paths on (M, h) in the presence of the cross gravitational wind $\mathbf{G}_\dagger$ as in* (8.1) *are the geodesics of the cross-slope metric F which satisfies*

$$||\mathbf{G}^T||_h^2 F^4 + 2\bar{g}\beta F^3 + (\alpha^2 - \bar{g}^2\beta^2)F^2 - 2\bar{g}\alpha^2\beta F - \alpha^4 = 0,$$

where $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ are given by (8.13) *and $||\mathbf{G}^T||_h < \frac{1}{2}$.*

We emphasize that F is a Finsler metric of general (α, β) type [155] and it gives rise to a natural and actual model of Finsler spaces as well as to a new application of this kind of Finsler metrics. Furthermore, we find the time geodesics of the cross-slope metric. With Theorem 8.1.1, all such solutions can be determined as follows.

[8] As it is shown on further reading, $||v||_h \in [1, \sqrt{5}/2)$.

Theorem 8.1.2. (Time geodesics) *Let the slope of a mountain be an n-dimensional Riemannian manifold (M,h), $n>1$, with the gravitational wind $\mathbf{G}^T$. The time-minimal paths on (M,h) in the presence of the cross wind $\mathbf{G}_\dagger$ are the time-parametrized solutions $\gamma(t)=(\gamma^i(t))$, $i=1,...,n$ of the ODE system*

$$\ddot{\gamma}^i(t)+2\mathcal{G}^i(\gamma(t),\dot{\gamma}(t))=0, \tag{8.3}$$

where

$$\begin{aligned}\mathcal{G}^i(\gamma(t),\dot{\gamma}(t)) &= \mathcal{G}^i_\alpha(\gamma(t),\dot{\gamma}(t))+\left[\tilde{\Theta}(r_{00}+2\alpha^2\tilde{R}r)+\alpha\tilde{\Omega}r_0\right]\frac{\dot{\gamma}^i(t)}{\alpha}\\ &\quad-\left[\tilde{\Psi}(r_{00}+2\alpha^2\tilde{R}r)+\alpha\tilde{\Pi}r_0\right]\frac{w^i}{\bar{g}}-\tilde{R}w^i{}_{|j}\frac{\alpha^2w^j}{\bar{g}^2},\end{aligned}$$

with

$$\begin{aligned}&\mathcal{G}^i_\alpha(\gamma(t),\dot{\gamma}(t))=\tfrac{1}{4}h^{im}\left(2\tfrac{\partial h_{jm}}{\partial\gamma^k}-\tfrac{\partial h_{jk}}{\partial\gamma^m}\right)\dot{\gamma}^j(t)\dot{\gamma}^k(t), \qquad \tilde{R}=-\tfrac{\bar{g}^2}{2\alpha^4\tilde{B}},\\ &r_{00}=-\tfrac{1}{\bar{g}}w_{j|k}\dot{\gamma}^j(t)\dot{\gamma}^k(t), \qquad r_0=\tfrac{1}{\bar{g}^2}w_{j|k}\dot{\gamma}^j(t)w^k, \qquad r=-\tfrac{1}{\bar{g}^3}w_{j|k}w^jw^k,\\ &\tilde{\Theta}=\tfrac{\bar{g}\alpha}{2\tilde{E}}(\alpha^6\tilde{A}\tilde{B}^2-\bar{g}\beta), \qquad \tilde{\Omega}=-\tfrac{\bar{g}^2}{\tilde{B}\tilde{E}}[\alpha^4\tilde{B}^3+\bar{g}\beta\tilde{B}+||\mathbf{G}^T||^2_h(\tilde{B}-\tilde{A})],\\ &\tilde{\Psi}=\tfrac{\bar{g}^2\alpha^2}{2\tilde{E}}(\alpha^4\tilde{A}^2\tilde{B}+1), \qquad \tilde{\Pi}=-\tfrac{\bar{g}^3}{2\tilde{B}\tilde{E}\alpha^3}[2\alpha^4\tilde{B}(\alpha^2\tilde{A}\tilde{B}-1)-\bar{g}\beta(\alpha^2\tilde{B}+1)],\\ &\tilde{A}=\tfrac{1}{\alpha^2}(\bar{g}\beta+\alpha^2-1), \qquad \tilde{B}=\tfrac{1}{\alpha^2}(2\bar{g}\beta+2\alpha^2-1), \quad \tilde{C}=\tfrac{1}{\alpha}\left(\alpha^2\tilde{B}+\bar{g}\beta\tilde{A}\right),\\ &\tilde{E}=\tilde{B}\tilde{C}^2\alpha^6+(||\mathbf{G}^T||^2_h\alpha^2-\bar{g}^2\beta^2)(\alpha^4\tilde{A}^2\tilde{B}+1)\end{aligned} \tag{8.4}$$

and $\alpha=\alpha(\gamma(t),\dot{\gamma}(t))$, $\beta=\beta(\gamma(t),\dot{\gamma}(t))$.

The proofs of the aforementioned results are presented in the next section.

8.2 Proofs of the main results

Setting the navigation problem on a slope of a mountain represented by an n-dimensional Riemannian manifold (M,h), $n>1$ and influenced by a cross gravitational wind $\mathbf{G}_\dagger$ given by (8.1), we provide the cross-slope metric with the necessary and sufficient conditions for its strong convexity as well as its time geodesics.

8.2.1 The cross-slope metric

The proof of Theorem 8.1.1 includes a sequence of lemmas which collect all requirments for a Finsler metric. The main tool in our approach is the gravitational wind $\mathbf{G}^T=-\bar{g}\omega^\sharp$, where $\bar{g}$ is the rescaled magnitude of the acceleration of gravity g (i.e. $\bar{g}=\lambda g$, $\lambda>0$), and $\omega^\sharp=h^{ji}\frac{\partial p}{\partial x^j}\frac{\partial}{\partial x^i}$ is the gradient vector field, with $p:M\to\mathbb{R}$ being a C^∞-function on M. For more details we refer the reader to [10, 20, 12].

Let u be the self-velocity and $u^\perp$ the orthogonal direction on u. By assuming the condition $||u||_h=1$ as it is usually done in the theoretical investigations on the Zermelo navigation (see e.g. [45]), one can observe that the resultant velocity is $v = u + \mathbf{G}_\dagger$, taking into account the effect of the cross gravitational wind. Within this geometrical framework two distinct ways can be followed in the proof of Theorem 8.1.1. One way is relied on the expression of $\mathbf{G}_\dagger$ as in (8.1), where $\mathbf{G}_{MAT}$ is the orthogonal projection of $\mathbf{G}^T$ on u. This allows us to arrive at the cross-slope metric in two steps, following the same technique as in Chapter 7 or [10]. First, to deform the background Riemannian metric by the vector field $-\mathbf{G}_{MAT}$, which is a direction-dependent deformation. In the second step, the resulting Finsler metric F (obtained in the first step) is deformed further by the vector field $\mathbf{G}^T$, i.e. a rigid translation, under the condition $F(x, -\mathbf{G}^T) < 1$ which guarantees that the walker on the mountainside can go forward in any direction; see [127] for more details.

Another way, which we will proceed below, is to use the fact that the cross gravitational wind $\mathbf{G}_\dagger$ coincides with the orthogonal projection of $\mathbf{G}^T$ on $u^\perp$ in this study and thus, it depends on the direction of the self-velocity u. Consequently, the indicatrix of the resulting cross-slope metric is the anisotropic deformation of the indicatrix of the background Riemannian metric by the vector field $\mathbf{G}_\dagger$.

Let θ be the angle between $\mathbf{G}^T$ and u. Roughly speaking, it represents the desired direction of motion and $\theta \in [0, 2\pi)$. It follows that

$$||\mathbf{G}_\dagger||_h = ||\mathbf{G}^T||_h |\sin\theta|, \text{ for any } \theta \in [0, 2\pi). \tag{8.5}$$

This clearly forces $||\mathbf{G}_\dagger||_h \leq ||\mathbf{G}^T||_h$. In particular, if $\theta \in \{0, \pi\}$, then $\mathbf{G}_\dagger = \overrightarrow{0}$ and $u = v$, or if $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, then $\mathbf{G}_\dagger = \mathbf{G}^T$.

Moreover, the cross gravitational wind $\mathbf{G}_\dagger$ can also be seen as the orthogonal projection of the resultant velocity v on $u^\perp$ because $v = u + \mathbf{G}_\dagger$. Denoting by ζ the angle between v and $u^\perp$, it follows that $||\mathbf{G}_\dagger||_h = ||v||_h \cos\zeta$, with $\zeta \in (0, \frac{\pi}{2}]$, and thus

$$h(v, \mathbf{G}_\dagger) = ||v||_h ||\mathbf{G}_\dagger||_h \cos\zeta = ||\mathbf{G}_\dagger||_h^2. \tag{8.6}$$

Note that $\zeta = \frac{\pi}{2}$ corresponds to the particular case $\theta \in \{0, \pi\}$. Now, taking into account (8.5) and (8.6), the relation $1 = ||u||_h = ||v - \mathbf{G}_\dagger||_h$ leads to

$$||v||_h = \sqrt{1 + ||\mathbf{G}^T||_h^2 \sin^2\theta}, \text{ for any } \theta \in [0, 2\pi). \tag{8.7}$$

Furthermore, it turns out that

$$\cos\zeta = \frac{||\mathbf{G}^T||_h |\sin\theta|}{||v||_h} \quad \text{and} \quad \sin\zeta = \frac{1}{||v||_h}. \tag{8.8}$$

Let $\tilde{\theta}$ be the angle between $\mathbf{G}^T$ and v. Then, $h(v, \mathbf{G}^T) = ||v||_h ||\mathbf{G}^T||_h \cos\tilde{\theta}$ and regarding θ, two cases are distinguished:

- If $\theta \in [0, \pi)$, then $\tilde{\theta} = \theta + \zeta - \frac{\pi}{2}$ and $h(v, \mathbf{G}^T) = ||v||_h ||\mathbf{G}^T||_h \sin(\theta + \zeta)$.
- If $\theta \in [\pi, 2\pi)$, then $\tilde{\theta} = \frac{\pi}{2} + \theta - \zeta$ and $h(v, \mathbf{G}^T) = -||v||_h ||\mathbf{G}^T||_h \sin(\theta - \zeta)$.

Making use of (8.8), both cases lead to

$$\sin^2\theta = \frac{||\mathbf{G}^T||_h^2 - \left[1 + h(v, \mathbf{G}^T) - ||v||_h^2\right]^2}{||\mathbf{G}^T||_h^2}.$$

This substituted into (8.7) yields $g_1(x,v)=0$, where

$$g_1(x,v)=||v||_h^4-\left[1+2h(v,\mathbf{G}^T)\right]||v||_h^2+h^2(v,\mathbf{G}^T)+2h(v,\mathbf{G}^T)-||\mathbf{G}^T||_h^2. \tag{8.9}$$

By applying Okubo's method [106], we obtain C^∞-functions $F(x,v)$ as the solutions of the equation $g_1(x,\frac{v}{F})=0$. This can be rewritten explicitly as a polynomial equation of degree four

$$||\mathbf{G}^T||_h^2F^4-2h(v,\mathbf{G}^T)F^3+\left[||v||_h^2-h^2(v,\mathbf{G}^T)\right]F^2+2||v||_h^2h(v,\mathbf{G}^T)F-||v||_h^4=0, \tag{8.10}$$

where F is evaluated at (x,v).

Now we are going to establish some properties of the functions $F(x,v)$, aiming to select a strongly convex Finsler metric among the roots of (8.10).

Lemma 8.2.1. *The functions $F(x,v)$, which satisfy* (8.10)*, are homogeneous of degree one with respect to v.*

Proof. On substituting $v=v^i\frac{\partial}{\partial x^i}$ with cv, $c>0$ into (8.10) we obtain

$$\begin{aligned}&||\mathbf{G}^T||_h^2F^4(x,cv)-2ch(v,\mathbf{G}^T)F^3(x,cv)+c^2[||v||_h^2-h^2(v,\mathbf{G}^T)]F^2(x,cv)\\&+2c^3||v||_h^2h(v,\mathbf{G}^T)F(x,cv)-c^4||v||_h^4=0.\end{aligned} \tag{8.11}$$

Differentiating (8.11) with respect to c and then setting $c=1$, we get

$$\left(\frac{\partial F}{\partial v^i}v^i-F\right)\varpi=0,$$

where F is evaluated at (x,v) and $\varpi=\left[h(v,\mathbf{G}^T)F-||v||_h^2\right]\left[F^2+h(v,\mathbf{G}^T)F-2||v||_h^2\right]$. If $\varpi\neq0$, then $\frac{\partial F}{\partial v^i}v^i=F$, i.e. F is positive homogeneous of degree one with respect to v. If $\varpi=0$, then $F=\frac{||v||_h^2}{h(v,\mathbf{G}^T)}$ or $F=\frac{1}{2}\left[-h(v,\mathbf{G}^T)\pm\sqrt{h^2(v,\mathbf{G}^T)+8||v||_h^2}\right]$. These are homogeneous of degree one with respect to v, but they do not check (8.10). ☐

Due to 8.2.1 and the fact that any nonzero $y\in T_xM$ can be expressed as $y=cv$, $c>0$, the extension of $F(x,v)$ to arbitrary nonzero vectors y, for any $x\in M$ is also homogeneous of degree one with respect to y and it satisfies the equation

$$\begin{aligned}&||\mathbf{G}^T||_h^2F^4(x,y)-2h(y,\mathbf{G}^T)F^3(x,y)+\left[||y||_h^2-h^2(y,\mathbf{G}^T)\right]F^2(x,y)\\&+2||y||_h^2h(y,\mathbf{G}^T)F(x,y)-||y||_h^4=0,\end{aligned} \tag{8.12}$$

with $F(x,v)=1$. Considering the notations

$$\alpha^2=||y||_h^2=h_{ij}y^iy^j\quad\text{and}\quad\beta=-\frac{1}{\bar{g}}h(y,\mathbf{G}^T)=h(y,\omega^\sharp)=b_iy^i, \tag{8.13}$$

$\alpha=\alpha(x,y)$, $\beta=\beta(x,y)$, $b=||\beta||_h=||\omega^\sharp||_h$ and $||\omega^\sharp||_h=\frac{1}{\bar{g}}||\mathbf{G}^T||_h$, (8.12) is equivalent to

$$||\mathbf{G}^T||_h^2F^4+2\bar{g}\beta F^3+(\alpha^2-\bar{g}^2\beta^2)F^2-2\bar{g}\alpha^2\beta F-\alpha^4=0, \tag{8.14}$$

where F is evaluated at (x,y). Thus, the indicatrix of F is defined by

$$I_F=\{(x,y)\in TM\mid(\alpha^2+\bar{g}\beta)^2-\alpha^2-2\bar{g}\beta-||\mathbf{G}^T||_h^2=0\}.$$

Lemma 8.2.2. *The equation* (8.14) *admits a unique positive root.*

Proof. Making use of the notation $\phi = \frac{F(x,y)}{\alpha}$ and dividing (8.14) by α^4, it can be expressed in the following form

$$||\mathbf{G}^T||_h^2\phi^4 + 2\bar{g}s\phi^3 + (1-\bar{g}^2s^2)\phi^2 - 2\bar{g}s\phi - 1 = 0, \tag{8.15}$$

where $\phi = \phi(||\mathbf{G}^T||_h^2, s)$ and $s = \frac{\beta}{\alpha}$. For the variable s we have $|s| \leq b = \frac{1}{\bar{g}}||\mathbf{G}^T||_h$ which results from the Cauchy-Schwarz inequality $|h(y,\omega^\sharp)| \leq ||y||_h||\omega^\sharp||_h$. Let $\phi^i = \phi^i(||\mathbf{G}^T||_h^2, s)$, $i = 1, ..., 4$ be the roots of (8.15). Among them there is at least one positive root because by the Viété relations, $\prod_{i=1}^{4} \phi^i < 0$, for any $s \in [-b, b]$. Moreover, basing on the Viété relations again, we get

$$\sum_{i=1}^{4}(\phi^i)^2 = -\frac{2}{||\mathbf{G}^T||_h^4}\left[||\mathbf{G}^T||_h^2 - \bar{g}^2s^2(2 + ||\mathbf{G}^T||_h^2)\right] \quad \forall s \in [-b, b],$$

which is negative for $s = 0$. Thus, at least for $s = 0$ there is a conjugate pair of complex roots among ϕ^i, $i = 1, ..., 4$. So, (8.15) admits at most two positive roots for any $s \in [-b, b]$. Since for $s = b$, (8.15) is reduced to $(\phi^2 - 1)(\phi||\mathbf{G}^T||_h + 1)^2 = 0$ and it allows only one positive root, we can conclude that among the roots ϕ^i there is a sole positive root, for any $s \in [-b, b]$. Consequently, there is also only a positive function $F(x, y) = \alpha\phi(||\mathbf{G}^T||_h^2, s)$, for any $s \in [-b, b]$, which satisfies (8.14) as claimed. □

Subsequently, we mean by $F(x, y) = \alpha\phi(||\mathbf{G}^T||_h^2, s)$ the unique positive root of (8.14), where $\phi(s, ||\mathbf{G}^T||_h^2) > 0$ is a C^∞-function provided by the sole positive root of (8.15), for any $s \in [-b, b]$. It is obvious that $\phi = \phi(s, ||\mathbf{G}^T||_h^2)$ is homogenous of degree zero with respect to y and thus, $F(x, y)$ is the general (α, β)-function with $b^2 = \frac{||\mathbf{G}^T||_h^2}{\bar{g}^2}$ as well as α and β are given by (8.13).

The last part of the proof of Theorem 8.1.1 refers to the necessary and sufficient conditions for strong convexity of the indicatrix I_F. One can proceed by applying Proposition 6.2.1. In order to make the argument work, however, we need the following lemma concerning some relations among derivatives of ϕ, with respect to s, which are then used to justify the positivity of some functions.

Lemma 8.2.3. *The function* ϕ *and its derivative with respect to* s, *i.e.* ϕ_2 *satisfy the following relations*

$$C\phi_2 = \bar{g}A\phi, \quad C(\phi - s\phi_2) = B, \quad B - 2A = \phi^2, \quad C\phi = B + \bar{g}sA\phi, \tag{8.16}$$

where

$$A = -\phi^2 + \bar{g}s\phi + 1, \quad B = -\phi^2 + 2\bar{g}s\phi + 2,$$
$$C = 2\ ||\mathbf{G}^T||_h^2\phi^3 + 3\bar{g}s\phi^2 + (1 - \bar{g}^2s^2)\phi - \bar{g}s. \tag{8.17}$$

Moreover, $C \neq 0$ *for any* $s \in [-b, b]$ *and*

$$\phi_2 = \frac{\bar{g}A}{C}\phi, \quad \phi - s\phi_2 = \frac{B}{C}, \quad \phi_{22} = \frac{\bar{g}^2}{C^3}(A^2B + \phi^4). \tag{8.18}$$

Proof. The main tool in the proof is (8.15) which is checked by ϕ, for any $s \in [-b, b]$. Its derivative with respect to s leads to the first relation in (8.16). Then, it immediately leads to the second identity from (8.16). The last two are justified by the notations (8.17) and (8.15).

Let us suppose by contradiction that there exists $s_0 \in [-b, b]$ such that $C(||\mathbf{G}^T||_h^2, s_0) = 0$. Under this assumption the first two relations in (8.16) imply that

$$A(||\mathbf{G}^T||_h^2, s_0) = B(||\mathbf{G}^T||_h^2, s_0) = 0,$$

and then, by the third one, we get $\phi(||\mathbf{G}^T||_h^2, s_0) = 0$. This contradicts ϕ being nonzero for any $s \in [-b, b]$.

The first two relations in (8.18) are coming from (8.16). The derivatives of the functions (8.17) with respect to s read

$$A_2 = \frac{\bar{g}}{C}(2A^2 + \phi^2), \qquad B_2 = \frac{2\bar{g}}{C}(AB + \phi^2), \qquad C_2 = \frac{\bar{g}}{C\phi}[-AB + (2 + \bar{g}s\phi)\phi^2] + 3\bar{g}A, \quad (8.19)$$

where $A_2 = \frac{\partial A}{\partial s}$, $B_2 = \frac{\partial B}{\partial s}$ and $C_2 = \frac{\partial C}{\partial s}$. These, along with $(\phi - s\phi_2)_2 = \frac{1}{C^2}(B_2 C - BC_2)$ or $\phi_{22} = \frac{\bar{g}}{C^2}(A_2 C + \bar{g}A^2 - AC_2)\phi$ lead to the last formula in (8.18). □

We can easily see that the functions A, B, C are homogenous of degree zero with respect to y because of the same homogeneity degree of ϕ. Moreover, some additional properties of these functions are essential for our aim.

Lemma 8.2.4. *The following assertions hold:*
i) $C(||\mathbf{G}^T||_h^2, s) > 0$, *for any* $s \in [-b, b]$;
ii) $B(||\mathbf{G}^T||_h^2, s) > 0$, *for any* $s \in [-b, b]$ *if and only if* $||\mathbf{G}^T||_h < \frac{1}{2}$.

Proof. i) By Lemma 8.2.3 we know that the C^∞-function C cannot be vanished on the interval $[-b, b]$. So, C has a constant sign on $[-b, b]$. Moreover, using (8.15), one easily proves that in the cases, where $s = \pm b$ we have

$$\phi(||\mathbf{G}^T||_h^2, \pm b) = 1, \quad A(||\mathbf{G}^T||_h^2, \pm b) = \pm ||\mathbf{G}^T||_h, \quad B(||\mathbf{G}^T||_h^2, \pm b) = 1 \pm 2||\mathbf{G}^T||_h.$$

These together with the last formula in (8.16) imply that

$$C(||\mathbf{G}^T||_h^2, \pm b) = (1 \pm ||\mathbf{G}^T||_h)^2 > 0$$

and thus, $C(||\mathbf{G}^T||_h^2, s) > 0$, for any $s \in [-b, b]$.
ii) We prove first that $B(||\mathbf{G}^T||_h^2, s) > 0$, for any $s \in [-b, b]$, under condition $||\mathbf{G}^T||_h < \frac{1}{2}$. Let us assume by contradiction that there exists $\tilde{s} \in [-b, b]$ such that $B(||\mathbf{G}^T||_h^2, \tilde{s}) = 0$. We are going to find $\tilde{s} \in [-b, b]$. On the one hand, if we put $s = \tilde{s}$ in (8.15), it is reduced to

$$2\,||\mathbf{G}^T||_h^2 \phi^2(||\mathbf{G}^T||_h^2, \tilde{s}) + 3\bar{g}\tilde{s}\phi(||\mathbf{G}^T||_h^2, \tilde{s}) + 1 = 0, \quad (8.20)$$

because $\phi(||\mathbf{G}^T||_h^2, \tilde{s}) > 0$ and $B(||\mathbf{G}^T||_h^2, \tilde{s}) = 0$. On the other hand, the second formula in (8.17) leads to $\phi(||\mathbf{G}^T||_h^2, \tilde{s}) = \bar{g}\tilde{s} + \sqrt{\bar{g}^2\tilde{s}^2 + 2}$. If we replace the latter formula in (8.20), it results $\tilde{s} = \pm\frac{1+4||\mathbf{G}^T||_h^2}{2\bar{g}\sqrt{3+4||\mathbf{G}^T||_h^2}}$, which contradicts $\tilde{s} \in [-b, b]$ due to the condition $||\mathbf{G}^T||_h < \frac{1}{2}$. Hence, $B(||\mathbf{G}^T||_h^2, s) \neq 0$, for any $s \in [-b, b]$ and moreover, it has a constant sign on $[-b, b]$. The fact that $B(||\mathbf{G}^T||_h^2, \pm b) = 1 \pm 2||\mathbf{G}^T||_h > 0$ implies that $B > 0$ on $[-b, b]$. The direct implication is obvious. □

An immediate consequence of Lemma 8.2.4 is that $\phi - s\phi_2 > 0$, for any $s \in [-b, b]$ if and only if $||\mathbf{G}^T||_h < \frac{1}{2}$. Also, by Lemma 8.2.4, if $||\mathbf{G}^T||_h < \frac{1}{2}$, then $\phi_{22} > 0$, for any $s \in [-b, b]$.

Remark 8.2.5. *Note that the force of the gravitational wind, restricted here by the inequality* $||\mathbf{G}^T||_h < \frac{1}{2}$ *can also be formulated via the variable s. More precisely,* $||\mathbf{G}^T||_h < \frac{1}{2}$ *if and only if* $|s| \leq b < \frac{1}{2\bar{g}}$. *Indeed, if* $||\mathbf{G}^T||_h < \frac{1}{2}$ *and making use of the Cauchy-Schwarz inequality, it follows that* $|s| \leq ||\omega^\sharp||_h = b = \frac{1}{\bar{g}}||\mathbf{G}^T||_h < \frac{1}{2\bar{g}}$. *The converse implication is trivial.*

Lemma 8.2.6. $\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0$ *if and only if* $|s| \leq b < \frac{1}{2\bar{g}}$.

Proof. According to the Cauchy-Schwarz inequality, one does have $|s| \leq b$. We assume that the inequality $\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0$ is checked, for any $s \in [-b, b]$. Then, for $s = -b$ it is reduced to $\frac{B(||\mathbf{G}^T||_h^2, -b)}{C(||\mathbf{G}^T||_h^2, -b)} > 0$, which gives $||\mathbf{G}^T||_h < \frac{1}{2}$. Thus, it results $|s| \leq b < \frac{1}{2\bar{g}}$ because of Lemma 8.2.5. Conversely, if $|s| \leq b < \frac{1}{2\bar{g}}$, then

$$\begin{aligned} \phi - s\phi_2 + (b^2 - s^2)\phi_{22} &= \frac{1}{C^3}\left[BC^2 + \bar{g}^2(b^2 - s^2)(A^2B + \phi^4)\right] \\ &> \frac{B}{C^3}\left[C^2 + \bar{g}^2(b^2 - s^2)A^2\right] \ > \ 0, \end{aligned}$$

because of Lemmas 8.2.4 and 8.2.5. □

Finally, by applying Proposition 6.2.1 it results that the unique positive root $F(x, y)$ of (8.14), i.e. the cross-slope metric is a Finsler metric if and only if $||\mathbf{G}^T||_h < \frac{1}{2}$. Thus, the indicatrix I_F is strongly convex if and only if $||\mathbf{G}^T||_h < \frac{1}{2}$. Besides, along any regular piecewise C^∞-curve γ on M one does have $F(\gamma(t), \dot{\gamma}(t)) = 1$, i.e. the time in which a walker goes along it, under the influence of the cross gravitational wind $\mathbf{G}_\dagger$. By recalling all the above results, the direction-dependent deformation of the background Riemannian metric h by $\mathbf{G}_\dagger$ provides the cross-slope metric which satisfies (8.14), with $||\mathbf{G}^T||_h < \frac{1}{2}$ and thus, Theorem 8.1.1 is justified. Furthermore, coming back to (8.7), the possible values of the resultant speed $||v||_h$ run through the interval $[1, \sqrt{5}/2)$, since $||\mathbf{G}^T||_h < \frac{1}{2}$.

8.2.2 The geodesics of the cross-slope metric

The proof of Theorem 8.1.2 is based on some technical computations which we split in two lemmas. Our goal is to arrive at the spray coefficients that correspond to the cross-slope metric F, and then the equations of the geodesics will be immediately provided. More specifically, even if we have only (8.14), which is satisfied by the cross-slope metric F, it is enough to find the time-minimal paths as the geodesics γ of F because $F(\gamma(t), \dot{\gamma}(t)) = 1$ along them. The claim of Theorem 8.1.2 is achieved by working with the cross-slope metric, which belongs to the class of general (α, β)-metrics and by employing the technique given by Proposition 6.2.2. We start by establishing some relations.

According to (8.13), for the cross-slope metric F the background Riemannian metric is h_{ij} and the differential 1-form β includes the gravitational wind $\mathbf{G}^T = -\bar{g}\omega^\sharp$, where $\omega^\sharp = h^{ji}\frac{\partial p}{\partial x^j}\frac{\partial}{\partial x^i}$ is the gradient vector field. With the notation $w_i = h_{ij}w^j$, where w^i denote the components of $\mathbf{G}^T$, it immediately results $w_i = -\bar{g}\frac{\partial p}{\partial x^i}$ and $\frac{\partial w_i}{\partial x^j} = \frac{\partial w_j}{\partial x^i}$. Moreover, in Lemma 7.2.6 or [10, Lemma 4.3] we have proved that β is closed, i.e. $s_{ij} = 0$ as well as the relations (7.39).

Taking into account Proposition 6.2.2 and the relations (8.18), the only thing left to do is to compute the derivatives ϕ_1 and ϕ_{12}.

Lemma 8.2.7. *The derivatives with respect to* $b^2 = \frac{||\mathbf{G}^T||_h^2}{\bar{g}^2}$ *and* s *of the function* ϕ, *i.e.* ϕ_1 *and* ϕ_{12} *hold the following relations*

$$\phi_1 = -\frac{\bar{g}^2}{2C}\phi^4, \quad \phi_{12} = -\frac{\bar{g}^3}{2C^3}\left[2AB + \bar{g}sA^2\phi - (2+\bar{g}s\phi)\phi^2\right]\phi^3. \tag{8.21}$$

Proof. Differentiating (8.15) with respect to $||\mathbf{G}^T||_h^2$, it results $\frac{\partial\phi}{\partial||\mathbf{G}^T||_h^2} = -\frac{1}{2C}\phi^4$. When substituted in $\phi_1 = \bar{g}^2\frac{\partial\phi}{\partial||\mathbf{G}^T||_h^2}$, this yields the first expression in (8.21). Moreover, the derivative of ϕ_1 with respect to s can be written in the form

$$\phi_{12} = -\frac{\bar{g}^2}{2C^2}(4C\phi^3\phi_2 - \phi^4 C_2).$$

Making use of the derivatives ϕ_2 and C_2 given by (8.18) and (8.19), the second formula in (8.21) follows at once from the latter relation. □

Lemma 8.2.8. *For the cross-slope metric* F *the spray coefficients* $\mathcal{G}^i$ *are related to the spray coefficients* $\mathcal{G}^i_\alpha = \frac{1}{4}h^{im}\left(2\frac{\partial h_{jm}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^m}\right)y^j y^k$ *of* α *by*

$$\mathcal{G}^i(x,y) = \mathcal{G}^i_\alpha(x,y) + \left[\Theta(r_{00} + 2\alpha^2 Rr) + \alpha\Omega r_0\right]\frac{y^i}{\alpha} - \left[\Psi(r_{00} + 2\alpha^2 Rr) + \alpha\Pi r_0\right]\frac{w^i}{\bar{g}} - \alpha^2 Rr^i, \tag{8.22}$$

with

$$\begin{aligned}
&r_{00} = -\tfrac{1}{\bar{g}}w_{i|j}y^i y^j, \quad r_0 = \tfrac{1}{\bar{g}^2}w_{i|j}w^j y^i, \quad r = -\tfrac{1}{\bar{g}^3}w_{i|j}w^i w^j, \quad r^i = \tfrac{1}{\bar{g}^2}w^i{}_{|j}w^j,\\
&R = -\tfrac{\bar{g}^2F^4}{2\alpha^4 B}, \quad \Theta = \tfrac{\bar{g}\alpha}{2EF}(\alpha^6AB^2 - \bar{g}\beta F^5), \quad \Psi = \tfrac{\bar{g}^2\alpha^2}{2E}(\alpha^4A^2B + F^4),\\
&\Omega = -\tfrac{\bar{g}^2F^2}{BE}[\alpha^4B^3 + \bar{g}\beta BF^3 + ||\mathbf{G}^T||_h^2(B-A)F^4],\\
&\Pi = -\tfrac{\bar{g}^3F^3}{2BE\alpha^3}[2\alpha^4B(\alpha^2AB - F^2) - \bar{g}\beta F^3(\alpha^2B + F^2)],
\end{aligned} \tag{8.23}$$

where

$$\begin{aligned}
&A = \tfrac{1}{\alpha^2}(-F^2 + \bar{g}\beta F + \alpha^2), \quad B = \tfrac{1}{\alpha^2}(-F^2 + 2\bar{g}\beta F + 2\alpha^2),\\
&C = \tfrac{1}{\alpha F}\left(\alpha^2B + \bar{g}\beta AF\right), \quad E = BC^2\alpha^6 + (||\mathbf{G}^T||_h^2\alpha^2 - \bar{g}^2\beta^2)(\alpha^4A^2B + F^4).
\end{aligned} \tag{8.24}$$

Proof. By Lemma 8.2.7 and the relations (8.18) a technical computation yields the following expressions

$$\begin{aligned}
&s\phi + (b^2 - s^2)\phi_2 = \tfrac{1}{\bar{g}C}(\bar{g}sB + ||\mathbf{G}^T||_h^2 A\phi),\\
&(\phi - s\phi_2)\phi_2 - s\phi\phi_{22} = \tfrac{\bar{g}}{C^3}(AB^2 - \bar{g}s\phi^5),\\
&\phi - s\phi_2 + (b^2 - s^2)\phi_{22} = \tfrac{1}{C^3}\left[BC^2 + (||\mathbf{G}^T||_h^2 - \bar{g}^2s^2)(A^2B + \phi^4)\right],\\
&(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22} = -\tfrac{\bar{g}^3}{2C^4}\left[2AB^2 - (2+\bar{g}s\phi)B\phi^2 - \bar{g}s\phi^5\right]\phi^3.
\end{aligned} \tag{8.25}$$

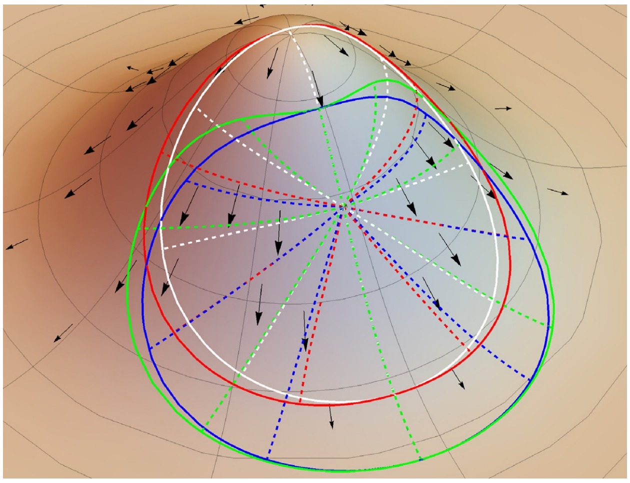


Figure 8.2: Time geodesics in a cross gravitational wind (dashed red) on $\mathfrak{G}$ compared to the Matsumoto case (dashed green), the Zermelo-Randers case (dashed blue) and the Riemannian case (dashed white). The corresponding unit time fronts are shown in solid colors and the gravitational wind $\mathbf{G}^T$ (black arrows) "blows" in the steepest downhill direction; $\bar{g} = 0.63$, $\Delta t = 1$, with a step of $\Delta\theta = \pi/4$ (8 paths in each case) and the initial point is positioned on the parallel of the strongest gravitational wind, i.e. $(\rho(0) = 1/\sqrt{2}, \varphi(0) = -\pi/4)$. It can be seen how the initial Riemannian geodesics and time front are deformed under the influence of $\mathbf{G}^T$, depending on the type of navigation problem.

Now, if we apply Proposition 6.2.2, taking into consideration the relations (7.39) along with the latter relations, Lemmas 8.2.3 and 8.2.7, where $\phi = \frac{F(x,y)}{\alpha}$, it results our claim. □

It should be mentioned that if $||\mathbf{G}^T||_h^2$ is constant, then the formula (8.22) is reduced to a expresion like in (7.43) with (8.23) because $r^i = r = r_0 = 0$ in this particular case.

To end the proof of Theorem 8.1.2, owing to the system (6.2) and Lemma 8.2.8 with $F(\gamma(t), \dot{\gamma}(t)) = 1$, one can write the ODE system (8.3) which yields the shortest time trajectories $\gamma(t) = (\gamma^i(t))$, $i = 1, ..., n$ on the slope of a mountain under the action of a cross gravitational wind.

For the sake of clarity and comparison with the recent study on the generalization of the Matsumoto slope-of-a-mountain problem (presented in Chapter 7, [10]), it is preferable to present the new outcome related to the cross-slope problem with the use of the same two-dimensional model, namely Gaussian bell-shaped surface $\mathfrak{G}$ described by the two-dimensional Gaussian function $z = \frac{3}{2}e^{-(x^2+y^2)}$. Following Theorem 8.1.1, the indicatrix of the cross-slope metric F on the surface $\mathfrak{G}$ is strongly convex if and only if $\bar{g} < \frac{\sqrt{2e+9}}{6} \approx 0.64$, being the same condition as in the standard Matsumoto problem.

Therefore, owing to Theorem 8.1.2 and Lemma 8.2.8, the time geodesics $\gamma(t) = (\rho(t), \varphi(t))$ in the cross wind on the slope $\mathfrak{G}$ are the solutions of the ODE system

$$\begin{cases} 0 = \ddot{\rho} + \frac{\rho}{9\rho^2 e^{-2\rho^2}+1}\left[9(1-2\rho^2)e^{-2\rho^2}\dot{\rho}^2 - \dot{\varphi}^2\right] + 2\left[\tilde{\Theta}(r_{00} + 2\alpha^2\tilde{R}r) + \alpha\tilde{\Omega}r_0\right]\frac{\dot{\rho}}{\alpha} \\ \quad -\frac{6\rho e^{-\rho^2}}{9\rho^2 e^{-2\rho^2}+1}\left[\tilde{\Psi}(r_{00} + 2\alpha^2\tilde{R}r) + \alpha\tilde{\Pi}r_0\right] - \frac{18\rho(1-2\rho^2)e^{-2\rho^2}}{(9\rho^2 e^{-2\rho^2}+1)^3}\alpha^2\tilde{R} \\ 0 = \ddot{\varphi} + \frac{2}{\rho}\dot{\rho}\dot{\varphi} + 2\left[\tilde{\Theta}(r_{00} + 2\alpha^2\tilde{R}r) + \alpha\tilde{\Omega}r_0\right]\frac{\dot{\varphi}}{\alpha} \end{cases},$$

where $\tilde{\Theta}$, $\tilde{R}$, $\tilde{\Omega}$, $\tilde{\Pi}$ and $\tilde{\Psi}$ are given by (8.4), $||\mathbf{G}^T||_h = \frac{3\bar{g}\rho e^{-\rho^2}}{\sqrt{9\rho^2 e^{-2\rho^2}+1}}$, $\bar{g} < \frac{\sqrt{2e+9}}{6} \approx 0.64$, $\rho = \rho(t)$, $\varphi = \varphi(t)$, with (7.51) and (7.53).

The outcome is presented in Figure 8.2, where the cross-slope time geodesics are compared to the unperturbed Riemannian paths, the standard Matsumoto geodesics and the Zermelo-Randers geodesics under the gravitational wind $\mathbf{G}^T$ with $\bar{g} = 0.63$. Moreover, the corresponding unit time fronts are shown in the respective solid colors. It is worthwhile to mention that the time front related to the cross-slope metric crosses both the Matsumoto and Zermelo isochrones. However, it contains the Riemannian one, bearing a similarity to that of MAT - ZNP correspondence, but the center lines of the limaçons (red, green) point in the opposite directions again. We obtained the Matsumoto and Zermelo paths, based on the slippery slope theory (Chapter 7 or [10]), where the cross-traction coefficient is equal to 1 or 0 therein, respectively.

8.3 Model of a slippery cross slope under gravitational wind

The previous results [11, 10] have been encouraging enough to merit further investigation. Continuing the above line of research naturally led us to the new model of a slippery cross slope presented below.

8.3.1 Slippery cross slope

In order to gain some intuition, we consider the 2-dimensional model of the slope including the inclined planes in what follows, while the general purely geometric solution to the time-optimal navigation problem described is valid for an arbitrary dimension.

Let us observe that actually each of two orthogonal components of gravitational wind, i.e. $\mathrm{Proj}_u\mathbf{G}^T$, $\mathrm{Proj}_{u^\perp}\mathbf{G}^T$ can be reduced partially due to traction, making use of a real parameter, and not only entirely like in [106] (the lateral one) or [11] (the longitudinal one). As described in Chapter 7, this has already been done in the case of the transverse component in a slippery slope model, where the cross-traction coefficient η runs through the interval $[0, 1]$, linking MAT and ZNP [10]. By analogy to such compensation of $\mathbf{G}^T$, we aim at considering a slippery slope model in the current study, however concerning the along-gravity scaling and introducing another parameter called an *along-traction coefficient* $\tilde{\eta} \in [0, 1]$. We assume that, while the Earth's gravity impacts a walker or a craft on the slope, the cross wind being perpendicular to a desired direction of motion u is regarded to act always entirely, whereas the effective wind, which pushes the craft downwards, can be compensated as depending on traction. In other words, the proposed model refers to a mountain slope, fixing the maximum cross-track additive continuously, for any direction of motion θ and gravity force $||\mathbf{G}^T||_h$ and admitting the along-track changes at the same time (longitudinal sliding). Consequently, the corresponding Finslerian indicatrix in the new setting will be based on the direction-dependent deformation of the background Riemannian metric h again. However, unlike all the preceding problems listed above, the equations of motion in the general form will now be

$$v_{\tilde{\eta}} = u + \mathrm{Proj}_{u^\perp}\mathbf{G}^T + (1 - \tilde{\eta})\mathrm{Proj}_u\mathbf{G}^T. \tag{8.26}$$

Thus, we can say that the influences of both components of gravitational wind are now somewhat reversed in comparison to the slippery slope investigated in [10]. To simplify the writing and to be in agreement with our previous notation, we will write $\mathbf{G}_{MAT}$ for $\mathrm{Proj}_u\mathbf{G}^T$, and

$\mathbf{G}_{MAT}^{\perp}$ for $\mathrm{Proj}_{u^{\perp}}\mathbf{G}^T$. The new model of the mountain slope is called a *slippery cross slope* and the related time-minimal navigation generalizes or complements previous investigations.

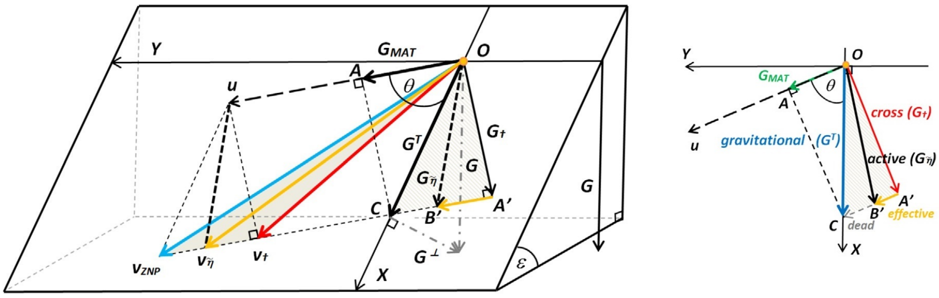


Figure 8.3: Left: A model of a planar slippery cross slope M under gravity $\mathbf{G} = \mathbf{G}^T + \mathbf{G}^{\perp}$, being analogous to the slippery slope as in Figure 7.1. Now the resultant velocity is represented by $v_{\tilde{\eta}}$ (yellow, $\tilde{\eta} \in [0,1]$) including the boundary cases for $\tilde{\eta} \in \{0,1\}$, i.e. the Zermelo case (v_{ZNP}, blue) and the cross slope case ($v_{\dagger}$, red), respectively. The longitudinal component $\overrightarrow{A'B'}$ of the active wind $\overrightarrow{OB'}$ (denoted by $\mathbf{G}_{\tilde{\eta}}$) w.r.t. u depends in particular on the along-traction coefficient $\tilde{\eta}$. Right: A gravitational wind and its decompositions on the slippery cross slope M, where $OA \perp OA'$. The gravitational wind $\mathbf{G}^T$ is a vector sum of an active wind and a dead wind ($\overrightarrow{B'C}$). The former is in turn decomposed into two orthogonal components: an effective wind $\overrightarrow{A'B'}$ and a cross wind $\overrightarrow{OA'} = \mathbf{G}_{\dagger}$ (which coincides with $\mathrm{Proj}_{u^{\perp}}\mathbf{G}^T$ in this case). The along-gravity force in general varies from 0 to $||\mathbf{G}_{MAT}||_h$ on the slippery cross slope, while the cross-gravity effect is always full, i.e. equal to $||\mathrm{Proj}_{u^{\perp}}\mathbf{G}^T||_h$. The direction θ of "unperturbed" motion indicated by the Riemannian velocity u (the control vector, dashed black) is measured clockwise from OX, where $||u||_h = 1$.

From (8.26) it follows manifestly that the self-velocity u is perturbed now by $\mathbf{G}_{\tilde{\eta}}$. Hence, the resultant velocity is expressed as[9]

$$v_{\tilde{\eta}} = u + \mathbf{G}_{\tilde{\eta}}, \tag{8.27}$$

where the active wind reads $\mathbf{G}_{\tilde{\eta}} = \mathbf{G}_{MAT}^{\perp} + (1-\tilde{\eta})\mathbf{G}_{MAT}$, where the former component stands for cross wind and the latter for effective wind. An equivalent formulation of the last relation is $\mathbf{G}_{\tilde{\eta}} = \tilde{\eta}\mathbf{G}_{\dagger} + (1-\tilde{\eta})\mathbf{G}^T$, since the cross wind is "blowing" with maximum force, so $\mathbf{G}_{\dagger} = \mathbf{G}_{MAT}^{\perp}$ in this case. Moreover, it follows evidently from the above that

$$\mathbf{G}_{\tilde{\eta}} = -\tilde{\eta}\mathbf{G}_{MAT} + \mathbf{G}^T, \tag{8.28}$$

where the component $\tilde{\eta}\mathbf{G}_{MAT}$ represents the dead wind. In particular, it is reasonable to expect that the edge cases, i.e. $\tilde{\eta} = 1$ and $\tilde{\eta} = 0$, will describe now, respectively, the cross-slope navigation (the action of maximum cross wind and minimum effective wind [10]), i.e. $\mathbf{G}_{\tilde{\eta}} = \mathbf{G}_{MAT}^{\perp}$, and the Zermelo navigation (the action of maximum both cross and effective winds), i.e. $\mathbf{G}_{\tilde{\eta}} = \mathbf{G}^T$. For the sake of clarity, see Figure 8.3.

Furthermore, it can be seen that the slippery slope problem with the cross-traction coefficient $\eta = 0$ from [10] and the current investigation with the new along-traction coefficient

[9]For brevity, we shall drop the subscript $\tilde{\eta}$ on $v_{\tilde{\eta}}$ when confusion is unlikely.

[10]For clarity's sake, see Figure 8.3, where $\overrightarrow{OA} = \mathrm{Proj}_u\mathbf{G}^T$ is in general the maximum effective wind, and $\overrightarrow{OA'} = \mathrm{Proj}_{u^{\perp}}\mathbf{G}^T$ the maximum cross wind, for any given θ and $||\mathbf{G}^T||_h$. A component of the gravitational wind $\mathbf{G}^T$ acts in full force if it is not reduced (partially or entirely), e.g. due to traction or drag.

$\tilde{\eta} = 0$ should coincide. Then this means that the scenario like in Zermelo's navigation under weak gravitational wind $\mathbf{G}^T$ will be located somewhat right in the middle between both approaches pieced together. It also stand naturally for the boundary and meeting case of the slippery slope and slippery-cross-slope solutions, where the gravitational wind "blows" in full[11] on the mountain slope, and the time geodesics come from Finsler metric of Randers type.

Finally, the current model of a slippery slope under cross gravitational wind complements the preceding investigations on the slope-of-a-mountain problems in a natural way. Namely, this fills in a missing part regarding the compensation of the along-gravity effect concerning the direction of motion indicated by the velocity u. As we will see on further reading, the obtained strong convexity conditions, being the basis for desired optimality of the trajectories on the slope, differ significantly from all those of the preceding navigation problems discussed above. In particular, for some $\tilde{\eta}$, it is admitted the norm of gravitational wind $||\mathbf{G}^T||_h$ to be greater than $||u||_h = 1$. For comparison, recall that the convexity condition in the Zermelo navigation, i.e. for a Randers metric is $||\mathbf{G}^T||_h < 1$ [45] and the corresponding conditions in both the Matsumoto and cross-slope metrics are most restrictive among all others, i.e $||\mathbf{G}^T||_h < 1/2$ [10, 11]. Moreover, in contrast to the situation on the slippery slope investigated in Chapter 7 or [10], the behaviour of the Finslerian indicatrix (time front), being now subject to along-traction expressed by $\tilde{\eta}$, is quite different. For instance, the new indicatrix crosses both edge cases (ZNP and CROSS) in two fixed points, while the parameter $\tilde{\eta}$ is running through the interval $(0, 1)$. This is shown in the presented example with an inclined plane in [12].

8.3.2 Statement of the main results

The problem of time-optimal navigation on a slippery slope under the cross-gravity effect (S-CROSS for short) can be posed as follows

> *Suppose a craft or a vehicle goes on a horizontal plane at maximum constant speed, while gravity acts perpendicularly on this plane. Imagine the craft moves now on a slippery cross slope of a mountain, with a given along-traction coefficient and under gravity. What path should be followed by the craft to get from one point to another in the minimum time?*

Our first goal is to provide the Finsler metric which serves as the solving tool for the S-CROSS problem. More precisely, posing the navigation problem on a slippery slope of a mountain represented by an n-dimensional Riemannian manifold (M, h), $n > 1$, under the action of the active wind $\mathbf{G}_{\tilde{\eta}}$ given by (8.28) here, supplies the slippery-cross-slope metrics as well as the necessary and sufficient conditions for their strong convexity. As in [10, 20, 11], the gravitational wind $\mathbf{G}^T = -\bar{g}\omega^\sharp$ turns out to be the main tool in our study, where $\bar{g}$ is the rescaled magnitude of the acceleration of gravity g (i.e. $\bar{g} = \lambda g$, $\lambda > 0$), and $\omega^\sharp = h^{ji}\frac{\partial p}{\partial x^j}\frac{\partial}{\partial x^i}$ is the gradient vector field, where $p : M \to \mathbb{R}$ is a C^∞-function on M.

Let u be the self-velocity of a moving craft on the slope. We assume $||u||_h = 1$, as standard in most theoretical investigations on the Zermelo navigation (see e.g. [45]). Taking into account the effect of the active wind $\mathbf{G}_{\tilde{\eta}}$, the resultant velocity $v_{\tilde{\eta}} = u + \mathbf{G}_{\tilde{\eta}}$ allows us to describe the slippery-cross-slope metric. A crucial role in our study is also played by the active wind, because it can be expressed as $\mathbf{G}_{\tilde{\eta}} = -\tilde{\eta}\mathbf{G}_{MAT} + \mathbf{G}^T$, $\tilde{\eta} \in [0, 1]$, where $\mathbf{G}_{MAT}$ is the orthogonal projection of $\mathbf{G}^T$ on u. It is worth mentioning that in this geometric context

[11] Both effective and cross winds are maximal in the Zermelo case, for any direction θ and wind force $||\mathbf{G}^T||_h$.

only the gravitational wind is known, the vector field $\mathbf{G}_{MAT}$ depends on direction of the self-velocity. In order to carry out the slippery-cross-slope metric, we conveniently divided the study into two steps including a sequence of cases and lemmas [12, Lemmas 3.1-3.3]. The first step describes deformation of the background Riemannian metric by the vector field $-\tilde{\eta}\mathbf{G}_{MAT}$, which is a direction-dependent deformation. The second step is mostly based on the resulting Finsler metric F of Matsumoto type, afforded by the first step. This is deformed by the gravitational vector field $\mathbf{G}^T$, i.e. a rigid translation, under the condition $F(x, -\mathbf{G}^T) < 1$ which practically ensures that a craft on the mountainside can go forward in any direction (for more details see [127]). Since the procedure follows more or less the same technique as in [10], which we already presented in detail in Chapter 7 (Step I and Step II), we omit to expand it here. Certainly, we have the following results.

Theorem 8.3.1. (Slippery-cross-slope metric) *Let a slippery cross slope of a mountain be an n-dimensional Riemannian manifold (M, h), $n > 1$, with the along-traction coefficient $\tilde{\eta} \in [0, 1]$ and the gravitational wind $\mathbf{G}^T$ on M. The time-minimal paths on (M, h) under the action of an active wind $\mathbf{G}_{\tilde{\eta}}$ as in* (8.28) *are the geodesics of the slippery-cross-slope metric $\tilde{F}_{\tilde{\eta}}$ which satisfies*

$$\tilde{F}_{\tilde{\eta}}\sqrt{\alpha^2 + 2\bar{g}\beta\tilde{F}_{\tilde{\eta}} + ||\mathbf{G}^T||_h^2\tilde{F}_{\tilde{\eta}}^2} = \alpha^2 + (2-\tilde{\eta})\bar{g}\beta\tilde{F}_{\tilde{\eta}} + (1-\tilde{\eta})||\mathbf{G}^T||_h^2\tilde{F}_{\tilde{\eta}}^2, \tag{8.29}$$

with $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ given by (8.13)*, where either $\tilde{\eta} \in [0, \frac{1}{3}]$ and $||\mathbf{G}^T||_h < \frac{1}{1-\tilde{\eta}}$, or $\tilde{\eta} \in (\frac{1}{3}, 1]$ and $||\mathbf{G}^T||_h < \frac{1}{2\tilde{\eta}}$. In particular, if $\tilde{\eta} = 1$, then the slippery-cross-slope metric yields the cross-slope metric, and if $\tilde{\eta} = 0$, then it is the Randers metric which solves the Zermelo navigation problem on a Riemannian manifold under a gravitational wind $\mathbf{G}^T$.*

The proof of Theorem 8.3.1 is based on all results obtaind in the aforementioned steps. For more detais we refer the reader to [12]. In addition, S-CROSS, which provides the slippery-cross-slope metric $\tilde{F}_{\tilde{\eta}}$ by (8.29), leads to a new application and a natural model of Finsler spaces with general (α, β) metrics [155].

The second goal is to find the time geodesics of the slippery-cross-slope metric. To do this, we exploit the geometrical and analytical properties, the main key being (8.29), and answering the above stated question this way. Thus, our second main result obtained in [12] is

Theorem 8.3.2. (Time geodesics) *Let a slippery cross slope of a mountain be an n-dimensional Riemannian manifold (M, h), $n > 1$, with the along-traction coefficient $\tilde{\eta} \in [0, 1]$ and the gravitational wind $\mathbf{G}^T$ on M. The time-minimal paths on (M, h) under the action of an active wind $\mathbf{G}_{\tilde{\eta}}$ as in* (8.28) *are the time-parametrized solutions $\gamma(t) = (\gamma^i(t))$, $i = 1, ..., n$ of the ODE system*

$$\ddot{\gamma}^i(t) + 2\tilde{\mathcal{G}}^i_{\tilde{\eta}}(\gamma(t), \dot{\gamma}(t)) = 0, \tag{8.30}$$

where

$$\begin{aligned}\tilde{\mathcal{G}}^i_{\tilde{\eta}}(\gamma(t), \dot{\gamma}(t)) &= \mathcal{G}^i_\alpha(\gamma(t), \dot{\gamma}(t)) + \left[\tilde{\Theta}(r_{00} + 2\alpha^2\tilde{R}r) + \alpha\tilde{\Omega}r_0\right]\frac{\dot{\gamma}^i(t)}{\alpha} \\ &\quad - \left[\tilde{\Psi}(r_{00} + 2\alpha^2\tilde{R}r) + \alpha\tilde{\Pi}r_0\right]\frac{w^i}{\bar{g}} - \tilde{R}w^i{}_{|j}\frac{\alpha^2 w^j}{\bar{g}^2},\end{aligned}$$

with

$$
\begin{aligned}
&\mathcal{G}^i_\alpha(\gamma(t),\dot\gamma(t)) = \tfrac{1}{4}h^{im}\left(2\tfrac{\partial h_{jm}}{\partial x^k} - \tfrac{\partial h_{jk}}{\partial x^m}\right)\dot\gamma^j(t)\dot\gamma^k(t), \qquad \tilde\Psi = \tfrac{\bar g^2\alpha^2}{2\tilde E}(\alpha^4\tilde A^2\tilde B + \tilde\eta^2),\\
&r_{00} = -\tfrac{1}{\bar g}w_{j|k}\dot\gamma^j(t)\dot\gamma^k(t), \qquad r_0 = \tfrac{1}{\bar g^2}w_{j|k}\dot\gamma^j(t)w^k, \qquad r = -\tfrac{1}{\bar g^3}w_{j|k}w^jw^k,\\
&\tilde R = \tfrac{\bar g^2}{2\alpha^4\tilde B}[(1-\tilde\eta)\,\alpha^2\tilde B - \tilde\eta], \qquad \tilde\Theta = \tfrac{\bar g\alpha}{2\tilde E}(\alpha^6\tilde A\tilde B^2 - \tilde\eta^2\bar g\beta),\\
&\tilde\Omega = \tfrac{\bar g^2}{\alpha^2\tilde B\tilde E}\{[(1-\tilde\eta)\,\alpha^2\tilde B - \tilde\eta](\alpha^6\tilde B^3 + \tilde\eta^2||\mathbf{G}^T||_h^2) - \tilde\eta^2\alpha^2(\bar g\beta\tilde B + ||\mathbf{G}^T||_h^2\tilde A)\},\\
&\tilde\Pi = \tfrac{\bar g^3}{2\alpha^3\tilde B\tilde E}\{[(1-\tilde\eta)\,\alpha^2\tilde B - \tilde\eta](2\alpha^6\tilde A\tilde B^2 - \tilde\eta^2\bar g\beta) + \tilde\eta^2\alpha^2\tilde B(2\alpha^2 + \bar g\beta)\},\\
&\tilde A = -\tfrac{1}{\alpha^2}\{\left[1-(2-\tilde\eta)\,(1-\tilde\eta)\,||\mathbf{G}^T||_h^2\right] - (2-\tilde\eta)^2\bar g\beta - (2-\tilde\eta)\alpha^2\},\\
&\tilde B = -\tfrac{1}{\alpha^2}\{[1-2(1-\tilde\eta)||\mathbf{G}^T||_h^2] - 2(2-\tilde\eta)\bar g\beta - 2\alpha^2\},\\
&\tilde C = \tfrac{1}{\alpha}\left(\alpha^2\tilde B + \bar g\beta\tilde A\right), \qquad \tilde E = \alpha^6\tilde B\tilde C^2 + (||\mathbf{G}^T||_h^2\alpha^2 - \bar g^2\beta^2)(\alpha^4\tilde A^2\tilde B + \tilde\eta^2)
\end{aligned}
\tag{8.31}
$$

and $\alpha = \alpha(\gamma(t),\dot\gamma(t))$, $\beta = \beta(\gamma(t),\dot\gamma(t))$, *and* w^i *denoting the components of* $\mathbf{G}^T$.

The proof of Theorem 8.3.2 comprises some technical computations which aim to reach the spray coefficients related to the slippery-cross-slope metric $\tilde F_{\tilde\eta}$. Once this is done, we can immediately supply the equations the time geodesics. Since the proof is similar in the spirit to that of Theorem 7.1.2, we omit it; for more details see [12, Section 4].

Finally, we deal with an example in dimension 2. For comparison and clarity, we continue the line of investigation presented presiously in Sections 7.2.3 and 8.2.2, considering the Gaussian bell-shaped hillside $\mathfrak{G}$ given by the Gaussian function $z = \frac{3}{2}e^{-(x^2+y^2)}$ (see also [10, 11, 12]). First we mention the result.

Lemma 8.3.3. *[12] The indicatrix of the slippery-cross-slope metric* $\tilde F_{\tilde\eta}$ *is strongly convex on the entire surface* $\mathfrak{G}$ *if and only if* $\bar g < \tilde\delta_2(\tilde\eta)$, *where*

$$
\tilde\delta_2(\tilde\eta) = \begin{cases} \frac{\sqrt{2e+9}}{3(1-\tilde\eta)}, & \text{if} \quad \tilde\eta \in [0, \frac{1}{3}] \\ \frac{\sqrt{2e+9}}{6\tilde\eta}, & \text{if} \quad \tilde\eta \in (\frac{1}{3}, 1] \end{cases}.
$$

Second, we show the $\tilde F_{\tilde\eta}$-geodesic equations, which are related to $\mathfrak{G}$. Owing to Theorem 8.3.2, the time geodesics $\gamma(t) = (\rho(t), \varphi(t))$ on the slippery cross slope of the surface $\mathfrak{G}$ are are provided by the solutions of the ODE system

$$
\begin{cases}
0 = \ddot\rho + \frac{\rho}{9\rho^2e^{-2\rho^2}+1}\left[9(1-2\rho^2)e^{-2\rho^2}\dot\rho^2 - \dot\varphi^2\right] + 2\left\{\tilde\Theta(r_{00} + 2\alpha^2\tilde R r) + \alpha\tilde\Omega r_0\right\}\frac{\dot\rho}{\alpha} \\
\quad -\frac{6\rho e^{-\rho^2}}{9\rho^2e^{-2\rho^2}+1}\left\{\tilde\Psi(r_{00} + 2\alpha^2\tilde R r) + \alpha\tilde\Pi r_0\right\} - \frac{18\rho(1-2\rho^2)e^{-2\rho^2}}{(9\rho^2e^{-2\rho^2}+1)^3}\alpha^2\tilde R \\
0 = \ddot\varphi + \frac{2}{\rho}\dot\rho\dot\varphi + 2\left\{\tilde\Theta(r_{00} + 2\alpha^2\tilde R r) + \alpha\tilde\Omega r_0\right\}\frac{\dot\varphi}{\alpha}
\end{cases},
$$

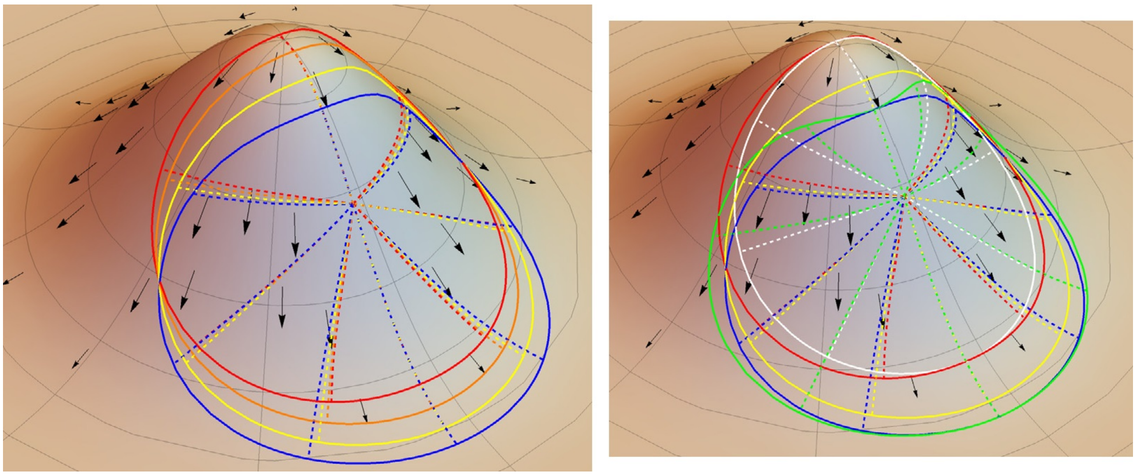


Figure 8.4: Left: the unit time fronts (solid colours) and the related time-minimizing geodesics (dashed colours; drawn with a step of $\Delta\theta = \pi/4$) on the slippery cross slope modelled by the rotational Gaussian bell-shaped surface $\mathfrak{G}$, under the action of gravitational wind, for various along-traction coefficients, i.e. $\tilde{\eta} \in \{$ 1/3 (yellow), 2/3 (orange), 0 (blue, the Zermelo case), 1 (red, the cross slope case)$\}$; $\bar{g} = 0.63$. The gravitational wind $\mathbf{G}^T$ (marked by black arrows) "blows" in the steepest downhill direction. The initial point is located on the parallel of the strongest gravitational wind, i.e. $(\rho(0) = 1/\sqrt{2}, \varphi(0) = -\pi/4)$. Right: the unit time fronts (solid colours) and the related time-minimizing geodesics (dashed colours) on the slippery cross slope as on the left, compared in addition to the Matsumoto (green) and Riemannian (white) cases; $\bar{g} = 0.63$. In particular, it can be observed how the initial (unperturbed) Riemannian geodesics and time front are deformed under the action of gravitational wind (marked by black arrows), depending on the along-traction coefficient $\tilde{\eta}$.

where $\tilde{\Theta}$, $\tilde{R}$, $\tilde{\Omega}$, $\tilde{\Pi}$ and $\tilde{\Psi}$ are given by (8.31), with $\mathbf{G}^T(\rho, \varphi) = \frac{3\bar{g}\rho e^{-\rho^2}}{9\rho^2 e^{-2\rho^2}+1}\frac{\partial}{\partial\rho}$, $\bar{g} < \tilde{\delta}_2(\tilde{\eta})$, together with (7.51) and (7.53), $\rho = \rho(t)$, $\varphi = \varphi(t)$.

Figure 8.4 (the left-hand side image) shows the slippery-cross-slope geodesics generated for various along-traction coefficients, i.e. $\tilde{\eta} \in \{0, 1/3, 2/3, 1\}$. Also, the corresponding unit time fronts are presented in solid colours. Moreover, the new solutions are compared to the Riemannian (white) and classical Matsumoto (green) geodesics as well as their fronts, under the action of the gravitational wind $\mathbf{G}^T$, where $\bar{g} = 0.63$ (see Figure 8.4, the right-hand side image).

Chapter 9

A general model for time-minimizing navigation on a mountain slope under gravity

In this chapter, based on the paper [13], we unify and extend all navigation problems, presented in Chapters 7 and 8, by a most general model of a slippery mountain slope, where both the transverse and longitudinal gravity-additives with respect to direction of motion are admitted to vary simultaneously in full ranges. The presented study focuses on finding optimal paths in the sense of time (time geodesics) in this general model of a slippery mountain slope under action of gravity.

9.1 Broader meaning of a slippery slope

We start by recalling the slope-of-a-mountain problem of Matsumoto (MAT) where the author [106] assumed that the slope is not slippery at all in the usual sense. The main objective was to find the time-minimizing paths on the mountain side under the influence of gravity. Such setting implied that the self-velocity u of a walker or a moving craft on the slope and the corresponding resultant velocity v always point in the same directions. In addition, the related speeds differ from each other by the entire along-gravity additive, being the norm of the orthogonal projection of the component of gravity, tangent to a slope on u, for any direction of motion. As is natural, there is higher resultant speed obtained in a downhill motion than in an uphill climbing, while the self-speed of a walker or a craft on the slope is kept maximum and constant. Thus, there is no drift (sliding) to any side taken into account. In other words, the cross component of a gravitational force pushing the walker off the u-track on the slope is always fully compensated (cancelled) in Matsumoto's model [106].

A more general approach in the context of time-minimizing solutions has been presented in Chapter 7 (based on [10]) describing a slippery slope model that admits the side drifts. This time the velocities u and v are not collinear in general whilst on the move, pointing in different directions. In that study (SLIPPERY for short) a *cross-traction coefficient* $\eta \in [0, 1]$ was introduced in particular by which the transverse effect (i.e. to the left or right side of the velocity u) of a gravitational force acting on a mountain slope was determined. Thus, in the boundary cases, the original Matsumoto problem on the non-slippery slope (in the usual sense) and the Zermelo navigation problem [157, 45] under a gravitational wind $\mathbf{G}^T$ are linked and generalized. Both become the particluar cases in the new setting, i.e. with $\eta = 1$ (no lateral drift, MAT) and $\eta = 0$ (maximum[1] lateral drift, ZNP), respectively.

Furthermore, to complement the exposition including the sliding effect, analogous model to the aforementioned has been presented in Section 8.3 (for details [10]), where the longitudinal drift with respect to a direction of motion indicated by the velocity u was taken into account in the equations of motion (S-CROSS for short). In this case, in turn, an *along-traction coefficient* $\tilde{\eta} \in [0,1]$ describes the range of another type of sliding, while the cross-gravity increment is always taken in full, i.e. $\eta = 0$, for any direction and gravity force. In this way it was possible to create a direct connection between the cross slope problem (CROSS for short, $\tilde{\eta} = 1$), presented in Chapter 8, and the classic Zermelo's navigation ($\tilde{\eta} = 0$) again. It is worth noting that ZNP is positioned somewhat right in the middle between both above approaches to modeling the slippery slopes pieced together, namely, SLIPPERY with scaling the lateral drift [10] and S-CROSS with the longitudinal compensation of the gravity impact on time-optimal motion on the mountain slope [12].

Both introduced parameters settle the type and range of compensation of the gravitational wind, however, varying individually, i.e. η cross the u-direction as considered in [10] and $\tilde{\eta}$ along the u-direction as studied in [12]. Those compensations determine next the transverse and longitudinal gravity-additives (slides) to own motion.

In the current investigation, we aim at analyzing the general case, admitting arbitrary type of a slide during the least time navigation on the slippery slope. This means that both traction coefficients η, $\tilde{\eta}$ will be included together in the general equations of motion. As a consequence, this study will generalize and collect the preceding results on time-optimal navigation under the action of gravity, which were obtained with a purely geometric approach by means of Finsler geometry, in particular in [10, 11, 12]. Moreover, the *slippery slope* will gain now a much broader meaning in the context of modeling time-minimizing motion on the hill side, as explained in the next subsection. The essential part of the study will refer to the strong convexity conditions, which correspond to the geometrically expressed conditions for optimality in the sense of time.

9.1.1 Navigation problems on a mountain slope with traction coefficients

Let us observe that by a pair of the traction coefficients it is possible to define in fact each *navigation problem* $\mathcal{P}$ in the slope model under action of gravity above mentioned, namely, $\mathcal{P}_{\eta,\tilde{\eta}} = (\eta, \tilde{\eta})$, where both parameters are fixed[2]. Thus, we have $\mathcal{P}_{MAT} = (1,0)$, $\mathcal{P}_{ZNP} = (0,0)$, $\mathcal{P}_{CROSS} = (0,1)$ and also $\mathcal{P}_{RIEM} = (1,1)$ which yields the Riemannian case, where the impact of gravity on motion is completely cancelled, i.e. $v = u$.

Furthermore, our objective is to present the general solution including all scenarios with the full ranges of both traction coefficients taken into account together, i.e. $\eta, \tilde{\eta} \in [0,1]$, and not just the boundary values ($\eta, \tilde{\eta} \in \{0,1\}$) as has been studied so far [106, 157, 45, 11]. This will lead to the new concrete problems on the slope like, for instance, $\mathcal{P} = (1/2, 1/3)$ or $\mathcal{P}' = (\pi/5, \sqrt{0.7})$, which in general have not been considered before. Consequently, any such setting will yield different type of motion on the slope, determined by given tractions. Then for any $\mathcal{P}_{\eta,\tilde{\eta}}$ the specific study leading to the time-optimal paths can be developed effectively,

[1] For any given direction of motion indicated by u and gravitational wind force $||\mathbf{G}^T||_h$.

[2] In general, the notation with both lower indices, i.e. $\mathcal{P}_{\eta\tilde{\eta}} = (\eta, \tilde{\eta})$ will be used especially for the slope problems that have not been specifically named like, e.g. MAT or ZNP.

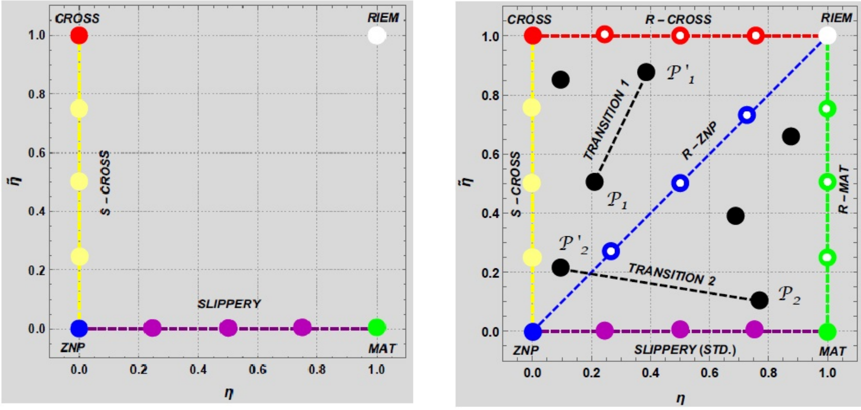


Figure 9.1: Left: The comparative presentation of the specific slope problems defined with the use of the traction coefficients $\eta, \tilde{\eta}$ on the problem diagram which have been studied in Riemann-Finsler geometry thus far. Right: The complete problem square diagram $\tilde{\mathcal{S}} = [0,1] \times [0,1]$ including all slippery slope problems under gravity with fixed (i.e. specific $\mathcal{P}$) and varying (i.e. transitions $\mathcal{T}_{\mathcal{P}}^{\mathcal{P}'}$) cross- and along-traction coefficients, where $\mathcal{P} = (\eta, \tilde{\eta})$, $\mathcal{P}' = (\eta', \tilde{\eta}') \in \tilde{\mathcal{S}}$.

creating the corresponding model for a potential application based on the arbitrary $(\eta, \tilde{\eta})$-navigation under the influence of gravity.

Moreover, in case at least one of the traction coefficients is varying (this means that the cross- or along-gravity additive is variable) as in SLIPPERY or S-CROSS so far we denote by $\mathcal{T}_{\mathcal{P}}^{\mathcal{P}'}$ (interchangeably $\mathcal{T}_{\eta,\tilde{\eta}}^{\eta',\tilde{\eta}'}$) a *transition* $\mathcal{T}$ between two specific slippery slope problems $\mathcal{P} = (\eta, \tilde{\eta})$ and $\mathcal{P}' = (\eta', \tilde{\eta}')$. Hence, $\mathcal{T}_{1,0}^{0,0}$ now describes SLIPPERY linking MAT and ZNP, with $\eta \in [0,1]$ and $\tilde{\eta} = 0$, as well as $\mathcal{T}_{0,1}^{0,0}$ represents S-CROSS linking CROSS and ZNP, with $\eta = 0$ and $\tilde{\eta} \in [0,1]$ [10, 12]. In such a way we can collect, compare and present all the above mentioned scenarios graphically in a clear and unified manner with the bird's eye view on the problem diagram in Figure 9.1. Actually, the figure also shows the state of the art in modeling time-optimal navigation on a slope of mountain under the action of gravity studied thus far (the left-hand side). More general, we aim at covering the cases in our solution, where the traction coefficients are running through the arbitrary intervals $I_\eta, I_{\tilde{\eta}} \subseteq [0,1]$ as well as the transitions $\mathcal{T}_{\mathcal{P}}^{\mathcal{P}'}$ that also connect the new type of problems $\mathcal{P}_{\eta,\tilde{\eta}}$ as above mentioned[3], e.g. $\mathcal{T}_{1/2,1/3}^{\pi/5,\sqrt{0.7}}$.

It is worth pointing out the meaning of "slippery" and "non-slippery" slope in the current context. As already emphasized, the initial Matsumoto's setting [106] was treated naturally as the non-slippery model in a usual sense. Roughly speaking, it is often adopted in the interpretations of various real world applications that the cross gravity effect is treated as somewhat "unwanted" or disturbing own (forward) motion indicated by the velocity u, while the along one is fully accepted. However, there are the situations in nature so that the approach can be exact opposite. It seems appropriate to mention the animals that move sideways, e.g. a sidewinder rattlesnake on a desert slope as well as the linear transverse ship's sliding motion side-to-side called sway on a dynamic surface of the sea in marine engineering

[3]Each fixed pair $(\eta, \tilde{\eta})$ yields a specific type of motion related to $\mathcal{P}_{\eta,\tilde{\eta}}$. In turn, the equation of motion are changing during transition. There is a certain analogy to a flight of a variable-sweep wing aircraft (a swing-wing design), modifying its geometry while flying.

or the description of the algebraic pedal curves and surfaces in geometry; see e.g. [11, 12, 64] in this regard. In our study, both types of gravity effects will be treated like having the same essence, without distingusihing which non-compensated gravity-additive (called a *slide* in our model) is "better" or "worse". This means that we can be dragged off forward-backward and sideways equally well on the slippery slope in the general model proposed.

Summarizing, the hitherto notion of a *slippery slope (problem)* is taking much broader meaning in comparison to the preceding studies [10, 12]. From now on, each $\mathcal{P} = (\eta, \tilde{\eta})$, $\eta, \tilde{\eta} \in [0, 1]$, defines in fact a different and specific navigation problem, where at least one type of gravity-increment (sliding) occurs. From such point of view only the Riemannian case, where the impact of gravitational wind is cancelled completely, i.e. $\eta = \tilde{\eta} = 1$ represents the non-slippery slope in the presence of gravity. Moreover, the current approach yields that even the slope in the classic Matsumoto model is considered as being slippery because of the longitudinal "slide" (i.e. the along-gravity gain) admitted, although it is construed as non-slippery at all in the usual sense.

Remark that in order to avoid any confusion with the first slippery slope model linking MAT and ZNP (i.e. $\eta \in [0, 1]$ and $\tilde{\eta} = 0$) which was introduced as a "slippery slope" in [10] and in Chapter 7 (called SLIPPERY herein) and its natural generalization in the current study, we slightly rename this particular case now as a *standard slippery slope*, referring to its meaning in the usual sense and to be in agreement with our previous terminology.

9.1.2 Problem formulation and main theorems

We can now formulate the main task to which the rest of this chapter is dedicated. Namely, the problem of time-minimizing navigation on a slippery mountain slope under the action of gravity is posed in the following way:

> *Suppose a walker, craft or a vehicle has a certain constant maximum speed as measured on a horizontal plane, while gravity acts perpendicular to this plane. Imagine now that the craft endeavours to move on a slippery slope of a mountain under gravity, admitting a traction-dependent sliding in arbitrary (downward) direction. What path should be followed by the craft to get from one point to another in the least time?*

In the general context of an n-dimensional Riemannian manifold with $\mathbf{G}^T = -\bar{g}\omega^\sharp$, where $\omega^\sharp$ is the gradient vector field and $\bar{g}$ is the rescaled gravitational acceleration g (see Section 9.2 and [10, 20, 11, 12]), we consider the active wind $\mathbf{G}_{\eta\tilde{\eta}}$, which represents the impact of gravity which is not compensated due to traction on the slippery slope, and defined by (9.11), with $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}} = [0, 1] \times [0, 1]$. Mention that it vanishes only when $\eta = \tilde{\eta} = 1$, i.e. the Riemannian case (RIEM for short). Let us also fix $\mathcal{S} = \tilde{\mathcal{S}} \setminus \{(1, 1)\}$.

The set $\tilde{\mathcal{S}}$ represents a complete problem square diagram for our exposition (see Figure 9.1, right-hand side). The solution to the posed problem is given by the new slippery slope metric in the general case, which is called $(\eta, \tilde{\eta})$-*slope metric*, as well as the corresponding time geodesics. Our main results are represented by the following two theorems.

Theorem 9.1.1. ($(\eta, \tilde{\eta})$-slope metric) *Let a slippery slope of a mountain be an n-dimensional Riemannian manifold (M, h), $n > 1$, with a cross-traction coefficient $\eta \in [0, 1]$, an along-traction coefficient $\tilde{\eta} \in [0, 1]$ and a gravitational wind $\mathbf{G}^T$ on M. The time-minimal paths on*

(M,h) under the action of an active wind $\mathbf{G}_{\eta\tilde{\eta}}$ as in (9.11) are the geodesics of an $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$, which satisfies

$$\tilde{F}_{\eta\tilde{\eta}}\sqrt{\alpha^2+2(1-\eta)\bar{g}\beta\tilde{F}_{\eta\tilde{\eta}}+(1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}_{\eta\tilde{\eta}}^2}=\alpha^2+(2-\eta-\tilde{\eta})\bar{g}\beta\tilde{F}_{\eta\tilde{\eta}}+(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2\tilde{F}_{\eta\tilde{\eta}}^2, \tag{9.1}$$

with $\alpha=\alpha(x,y)$, $\beta=\beta(x,y)$ given by (9.17), where either $||G^T||_h<\frac{1}{1-\tilde{\eta}}$ and $(\eta,\tilde{\eta})\in D_1\cup D_2$, or $||G^T||_h<\frac{1}{2|\eta-\tilde{\eta}|}$ and $(\eta,\tilde{\eta})\in D_3\cup D_4$, where

$$\mathcal{D}_1=\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\eta\geq\tilde{\eta}>2\eta-1\},\qquad \mathcal{D}_2=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\tfrac{3\tilde{\eta}-1}{2}<\eta<\tilde{\eta}\right\},$$

$$\mathcal{D}_3=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\eta\geq\tfrac{1}{2},\ \tilde{\eta}\leq 2\eta-1\right\},\qquad \mathcal{D}_4=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\tilde{\eta}\geq\tfrac{1}{3},\ \eta\leq\tfrac{3\tilde{\eta}-1}{2}\right\},$$

$S=\bigcup_{i=1}^{4}D_i$ and $D_i\cap D_j=\varnothing$, for any $i\neq j$, $i,j=1,...,4$. No restriction should be imposed on $||G^T||_h$ if $\eta=\tilde{\eta}=1$. In particular, a slope metric of type $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ is reduced to a Randers metric, a Matsumoto metric, a cross slope metric and a Riemannian metric h, respectively.

Moreover, an $(\eta,0)$-slope metric is a (standard) slippery slope metric $\tilde{F}_{\eta}$ and a $(0,\tilde{\eta})$-slope metric is a slippery-cross-slope metric $\tilde{F}_{\tilde{\eta}}$, both presented in the Chapters 7 and 8 ([10, Theorem 1.1] and [12, Theorem 1.1], respectively).

For clarity's sake, the partition of $\tilde{\mathcal{S}}$ into the mutually disjoint subsets $\mathcal{D}_i, i=1,...,4$ is illustrated in Figure 9.3. It is also worth mentioning that the above theorem now implies as the particular cases the solutions to: the original Matsumoto's slope-of-a-mountain problem (MAT), Zermelo's navigation problem (ZNP) on a Riemannian manifold under a gravitational wind $\mathbf{G}^T$ as well as CROSS. Furthermore, $\tilde{F}_{\eta\tilde{\eta}}$ provides a new Finsler metric of general (α,β) type.

The second theorem enables us to find time geodesics that correspond to an $(\eta,\tilde{\eta})$-slope metric, giving an answer to the research question posed above. Namely, we have obtained

Theorem 9.1.2. (Time geodesics) *Let a slippery slope of a mountain be an n-dimensional Riemannian manifold (M,h), $n>1$, with a cross-traction coefficient $\eta\in[0,1]$, an along-traction coefficient $\tilde{\eta}\in[0,1]$ and a gravitational wind $\mathbf{G}^T$ on M. The time-minimal paths on (M,h) under the action of an active wind $\mathbf{G}_{\eta\tilde{\eta}}$ as in (9.11) are the time-parametrized solutions $\gamma(t)=(\gamma^i(t))$, $i=1,...,n$ of the ODE system*

$$\ddot{\gamma}^i(t)+2\tilde{\mathcal{G}}^i_{\eta\tilde{\eta}}(\gamma(t),\dot{\gamma}(t))=0, \tag{9.2}$$

where

$$\begin{aligned}\tilde{\mathcal{G}}^i_{\eta\tilde{\eta}}(\gamma(t),\dot{\gamma}(t)) &= \mathcal{G}^i_{\alpha}(\gamma(t),\dot{\gamma}(t))+\left[\tilde{\Theta}(r_{00}+2\alpha^2\tilde{R}r)+\alpha\tilde{\Omega}r_0\right]\frac{\dot{\gamma}^i(t)}{\alpha}\\ &\quad-\left[\tilde{\Psi}(r_{00}+2\alpha^2\tilde{R}r)+\alpha\tilde{\Pi}r_0\right]\frac{w^i}{\bar{g}}-\tilde{R}w^i{}_{|j}\frac{\alpha^2w^j}{\bar{g}^2},\end{aligned}$$

with

$$\mathcal{G}^i_\alpha(\gamma(t),\dot\gamma(t)) = \frac{1}{4}h^{im}\left(2\frac{\partial h_{jm}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^m}\right)\dot\gamma^j(t)\dot\gamma^k(t), \qquad \tilde\Psi = \frac{\bar g^2\alpha^2}{2\tilde E}[\alpha^4\tilde A^2\tilde B + (\tilde\eta-\eta)^2],$$

$$r_{00} = -\frac{1}{\bar g}w_{j|k}\dot\gamma^j(t)\dot\gamma^k(t), \qquad r_0 = \frac{1}{\bar g^2}w_{j|k}\dot\gamma^j(t)w^k, \qquad r = -\frac{1}{\bar g^3}w_{j|k}w^jw^k,$$

$$\tilde R = \frac{(1-\eta)\bar g^2}{2\alpha^4\tilde B}[(1-\tilde\eta)\,\alpha^2\tilde B - (\tilde\eta-\eta)], \qquad \tilde\Theta = \frac{\bar g\alpha}{2\tilde E}[\alpha^6\tilde A\tilde B^2 - (\tilde\eta-\eta)^2\bar g\beta],$$

$$\tilde\Omega = \frac{(1-\eta)\bar g^2}{\alpha^2\tilde B\tilde E}\{[(1-\tilde\eta)\,\alpha^2\tilde B - (\tilde\eta-\eta)][\alpha^6\tilde B^3 + (\tilde\eta-\eta)^2||\mathbf{G}^T||_h^2] - (\tilde\eta-\eta)^2\alpha^2(\bar g\beta\tilde B + ||\mathbf{G}^T||_h^2\tilde A)\},$$

$$\tilde\Pi = \frac{(1-\eta)\bar g^3}{2\alpha^3\tilde B\tilde E}\{[(1-\tilde\eta)\,\alpha^2\tilde B - (\tilde\eta-\eta)][2\alpha^6\tilde A\tilde B^2 - (\tilde\eta-\eta)^2\bar g\beta] + (\tilde\eta-\eta)^2\alpha^2\tilde B[2\alpha^2 + (1-\eta)\bar g\beta]\},$$

$$\tilde A = -\frac{1}{\alpha^2}\{(1-\eta)\left[1-(2-\eta-\tilde\eta)(1-\tilde\eta)||\mathbf{G}^T||_h^2\right] - (2-\eta-\tilde\eta)^2\bar g\beta - (2-\eta-\tilde\eta)\alpha^2\},$$

$$\tilde B = -\frac{1}{\alpha^2}\{[1-2(1-\eta)(1-\tilde\eta)||\mathbf{G}^T||_h^2] - 2(2-\eta-\tilde\eta)\bar g\beta - 2\alpha^2\},$$

$$\tilde C = \frac{1}{\alpha}\left(\alpha^2\tilde B + \bar g\beta\tilde A\right), \qquad \tilde E = \alpha^6\tilde B\tilde C^2 + (||\mathbf{G}^T||_h^2\alpha^2 - \bar g^2\beta^2)[\alpha^4\tilde A^2\tilde B + (\tilde\eta-\eta)^2] \tag{9.3}$$

and $\alpha = \alpha(\gamma(t),\dot\gamma(t))$, $\beta = \beta(\gamma(t),\dot\gamma(t))$, *and* w^i *denoting the components of* G^T.

The chapter is structured in the following way. Our concern in Section 9.2 is focused on the general model of a slippery mountain slope under gravity, starting with some special navigation problems (so called the reduced ZNP, the reduced MAT and the reduced CROSS) which are achieved by the transition from the initial Riemannian background to the Zermelo's navigation problem (ZNP) under a weak gravitational wind $\mathbf{G}^T$, the Matsumoto's slope-of-a-mountain problem (MAT) and the cross slope problem (CROSS), respectively. Then, in Section 9.3.1 we perform the proof of Theorem 9.1.1, dividing it into two steps including a sequence of cases and lemmas. In Section 9.3.2 we prove Theorem 9.1.2 which is based on some technical results.

9.2 New models of a slippery mountain slope

Let (M,h) be an n-dimensional Riemannian manifold, $n>1$, which represents a model for a slippery slope of a mountain. Let $\omega^\sharp = h^{ji}\frac{\partial p}{\partial x^j}\frac{\partial}{\partial x^i}$ be the gradient vector field of p, where $p: M\to\mathbb{R}$ is a C^∞-function on M. Making use of $\omega^\sharp$, we have defined the gravitational wind $\mathbf{G}^T = -\bar g\omega^\sharp$, where $\bar g$ is the rescaled magnitude of the acceleration of gravity g (i.e. $\bar g = \lambda g$, $\lambda>0$) [10, 20, 11, 12].

Based on scaling, we assume throughout this section that we work with the self-velocity u of a moving craft on the slope and $||u||_h = \sqrt{h(u,u)} = 1$. Along this section we also refer to a 2-dimensional model for the slope, more precisely, to the inclined plane for a better view of the study, although the time-optimal navigation problems described in this work are valid for an arbitrary dimension.

Looking at the known cases visualized graphically in Figure 9.1, left-hand side, we first propose some new scenarios. Namely, we are going to consider three special transitions with varying traction, linking the Riemannian set-up with the Matsumoto, Zermelo and cross slope cases. As we can observe right below, it is possible to obtain the explicit form of the Finsler metrics in the first two problems, which are respectively of the Randers and Matsumoto type, including the parameter η or $\tilde\eta$ in their formulas in addition.

9.2.1 Some special cases

We begin by referring to the Zermelo navigation problem, where the purely geometric solution is given by a Finsler metric of Randers type, under the action of space-dependent weak vector field W on a Riemannian manifold (M, h) [127, 71, 45, 157].

Reduced ZNP

Let us take into account the setting $\eta = \tilde{\eta} \in [0, 1]$. In the sequel, this scenario is called the *reduced Zermelo navigation problem* (R-ZNP for short) and it is marked illustratively by the dashed blue diagonal in Figure 9.1, the right-hand side. Since the traction parameters run through the full range, we can connect the Riemannian and Zermelo cases directly by the transition $\mathcal{T}_{1,1}^{0,0}$ along the diagonal $\eta = \tilde{\eta}$. More precisely, the resultant velocity in this case is defined as follows

$$v_{R-ZNP} = u + (1-\eta)\mathbf{G}^T, \tag{9.4}$$

for any $\eta \in [0, 1]$. In other words, both cross and effective winds are compensated respectively by the cross- and along-traction coefficients at the same time and equally, i.e. $P \in OC$ in Figure 9.2, the right-hand side.

Now we are in position to apply the Zermelo's navigation technique as in [71, 45], with the navigation data (h, W), i.e. deforming the Riemannian metric h by a weak vector field $W = (1-\eta)\mathbf{G}^T$, i.e. $||W||_h = (1-\eta)||\mathbf{G}^T||_h < 1$. Thus, using the condition on the self-speed $||u||_h = 1$ and (9.4), we get $||v_{R-ZNP}||_h^2 - 2(1-\eta)h(v_{R-ZNP}, \mathbf{G}^T) - 1 + (1-\eta)^2||\mathbf{G}^T||_h^2 = 0$. This leads to the Finsler metric of Randers type $\tilde{F}_{R-ZNP}$ including either of the traction coefficients as a parameter. The result is

$$\tilde{F}_{R-ZNP}(x, y) = \frac{\sqrt{[(1-\eta)h(y, \mathbf{G}^T)]^2 + \lambda_\eta ||y||_h^2}}{\lambda_\eta} - \frac{(1-\eta)h(y, \mathbf{G}^T)}{\lambda_\eta}, \tag{9.5}$$

with $\lambda_\eta = 1 - (1-\eta)^2||\mathbf{G}^T||_h^2$, for any $(x, y) \in TM$. In particular, if $\eta = 0$, i.e. $P = C$ in Figure 9.2, then the last equation yields the standard Randers metric, which represents the solution to ZNP under the action of a full gravitational wind $\mathbf{G}^T$. On the other hand, for $\eta = 1$ we get the non-slippery Riemannian case, i.e. $P = O$ in Figure 9.2.

We notice that for every $\eta \in [0, 1]$ (along the diagonal $\eta = \tilde{\eta}$), the indicatrices of the Randers type metrics $\tilde{F}_{R-ZNP}$ are the "cloned" ellipsoids because by Zermelo's navigation, the Riemannian h-circle (ellipsoid) is only rigidly translated by $(1-\eta)\mathbf{G}^T$, under the restriction $(1-\eta)||\mathbf{G}^T||_h < 1$; see e.g. [71]. In particular, there is not any anisotropic deformation of the indicatrices and only rigid translation is applied, while transiting between two arbitrary problems in R-ZNP.

Reduced MAT

Now we set $\eta = 1$ and $\tilde{\eta} \in [0, 1]$, calling this scenario the *reduced Matsumoto slope-of-a-mountain problem* (R-MAT for short). The along-traction coefficient $\tilde{\eta}$ runs through the entire range, so RIEM and MAT can be linked directly by the transition $\mathcal{T}_{1,1}^{1,0}$ which is also included in the slippery slope model. This situation is indicated by the dashed green segment in Figure 9.1, the right-hand side. Thus, the equation of motion for this case reads

$$v_{R-MAT} = u + (1-\tilde{\eta})\mathbf{G}_{MAT}, \tag{9.6}$$

for any $\tilde{\eta} \in [0,1]$, where $\mathbf{G}_{MAT}$ is the orthogonal projection of $\mathbf{G}^T$ on u. Moreover, it follows immediately that $P \in OA$ in Figure 9.2, the right-hand side, so the cross wind is vanished, whilst the effective wind is scaled by the along-traction coefficient $\tilde{\eta}$. Since the velocities v_{R-MAT} and u are always collinear in this case, R-MAT is based on a direction-dependent deformation of the background Riemannian metric h by the vector field $(1-\tilde{\eta})\mathbf{G}_{MAT}$. By applying the navigation technique as in [71, 45], where $||v_{R-MAT}||_h = 1 \pm ||(1-\tilde{\eta})\mathbf{G}_{MAT}||_h$ ("+" for downhill and "-" for uphill motion), as well as [12, Step I], the resultant Finsler metric is obtained explicitly. More precisely, it is the (α,β)-metric of Matsumoto type including the along-traction coefficient $\tilde{\eta} \in [0,1]$ as a parameter, and denoted by $\tilde{F}_{R-MAT}$. We thus get

$$\tilde{F}_{R-MAT}(x,y) = \frac{||y||_h^2}{||y||_h + (1-\tilde{\eta})h(y,\mathbf{G}^T)}, \tag{9.7}$$

for any $(x,y) \in TM_0$, under the strong convexity restriction $(1-\tilde{\eta})||\mathbf{G}^T||_h < \frac{1}{2}$. In particular, if $\tilde{\eta} = 0$, then the last equation leads to the standard Matsumoto metric [106, 131], which stands for the solution to MAT, i.e. $P = A$ in Figure 9.2, the right-hand side. On the other edge, for $\tilde{\eta} = 1$ we get the Riemannian case, i.e. $P = O$ in Figure 9.2, the right-hand side, because the impact of the gravitational wind is then compensated completely during such kind of motion on the slope. In contrast to R-ZNP, there is not any rigid translation of the indicatrix of $\tilde{F}_{R-MAT}$ and only anisotropic deformation is applied, while transiting between two arbitrary problems included in R-MAT.

Reduced CROSS

As the last special problem mentioned in this subsection we consider the setting $\tilde{\eta} = 1$ and $\eta \in [0,1]$. By analogy to the cases described above, this scenario is named the *reduced cross slope problem* (R-CROSS for short) and indicated by the dashed red segment in Figure 9.1, the right-hand side. As the cross-traction coefficient η runs through the full range, RIEM and CROSS can be linked now by the transition $\mathcal{T}_{1,1}^{0,1}$, becoming the particular and edge cases in the current set-up. It follows from the above that the related equation of motion is[4]

$$v_{R-CROSS} = u + (1-\eta)\mathbf{G}_{MAT}^{\perp}, \tag{9.8}$$

for any $\eta \in [0,1]$. Hence, this yields $P \in OA'$ in Figure 9.2, the right-hand side and the effective wind is zeroed while the cross wind is varying, depending on cross-traction on the slippery mountain slope. In particular, if $\eta = 0$, i.e. $P = A'$ in Figure 9.2, the right-hand side, then the solution is given by a cross slope metric (Chapter 7 or [11]). On the other end, if $\eta = 1$, i.e. $P = O$, then we are led to the Riemannian metric h.

Having solved two previous problems explicitly, one may expect that the similar ease of investigation will be in the third analogous scenario. Unfortunately, the solution is much more complicated now. In contrast to R-MAT and R-ZNP, this time we do not get a "simple" explicit form of the Finsler metric. As shown on further reading, it is signficantly nontrivial and could be studied individually[5], however the corresponding solution can be extracted as the

[4] $\mathbf{G}_{MAT}^{\perp}$ is $\text{Proj}_{u^{\perp}}\mathbf{G}^T$, $\mathbf{G}_{MAT}^{\perp} = -\mathbf{G}_{MAT} + \mathbf{G}^T$ (i.e. $\overrightarrow{OA'}$ in 9.2) and recall that $\mathbf{G}_{MAT}$ stands for $\text{Proj}_u\mathbf{G}^T$ (i.e. $\overrightarrow{OA}$ in 9.2).

[5] We remark that the main proof concerning R-CROSS studied individually would follow the analogous way as in [11], however with the scaling factor $(1-\eta)$ for the vector field $\mathbf{G}_{MAT}^{\perp}$, included from the beginning in the related equations of motion. Similarly, the corresponding scaling factor in R-MAT is $(1-\tilde{\eta})$, however referring to the vector field $\mathbf{G}_{MAT}$, as well as $(1-\eta)$ with reference to the gravitational wind $\mathbf{G}^T$ in R-ZNP.

particular case from the general result in Section 9.3.1, having the strong convexity condition $||\mathbf{G}^T||_h < \frac{1}{2(1-\eta)}$, $\eta \in [0,1)$, in this case. Actually, the computational difficulty could have been expected in advance, since the detailed solution to CROSS has already been analyzed in [11] (here in Chapter 8) and it stands for the edge case in the current setting. We decided, however, to mention this new type of slope problem here so that the problem diagram form a square clearly after including the last missing side as shown in Figure 9.1, the right-hand side.

Unlike R-ZNP and R-MAT, the evolution of the indicatrix is based on both anisotropic deformation and rigid translation combined together, while transiting between two arbitrary problems included in R-CROSS.

To complete, all three cases described above start (or end) at the Riemannian vertex $(1,1)$, including the transition along the diagonal of the problem square diagram. One can also consider another transition along the second diagonal of $\tilde{\mathcal{S}}$, i.e. $\mathcal{T}_{1,0}^{0,1}$ linking MAT and CROSS. In this case, we have the relation $\tilde{\eta} = 1-\eta$, $\eta \in [0,1]$, and so $P \in AA'$ (Figure 9.2), where one traction coefficient is a non-identity[6] (linear) function of another. As in R-CROSS, the corresponding resultant metric cannot be explicitly obtained in a simple form.

9.2.2 General case

Following the above presented reasoning in a more general context one can ask whether the Riemannan case can also be linked with a problem $\mathcal{P}$ defined by a pair $(\eta, \tilde{\eta})$, indicating the interior of the square diagram $\tilde{\mathcal{S}}$, and not just its boundary $\partial\tilde{\mathcal{S}}$ (R-CROSS, SLIPPERY, R-MAT, R-CROSS) or one diagonal, i.e. $\tilde{\eta} = \eta$ (R-ZNP) as untill now. Moreover, we can actually look at the model even more generally, connecting two arbitrary problems $\mathcal{P} = (\eta, \tilde{\eta})$ and $\mathcal{P}' = (\eta', \tilde{\eta}')$ of the whole diagram, where $\eta, \eta', \tilde{\eta}, \tilde{\eta}' \in [0,1]$. Thus, we are going to enter the interior of $\tilde{\mathcal{S}}$ created by four cornered cases, i.e. RIEM, MAT, ZNP and CROSS (respectively, O, A, C and A' in Figure 9.2), the right-hand side, and ultimately to cover its whole area. From such point of view each $\mathcal{P}$ with the traction coefficients being fixed uniquely defines different and specific navigation problem on the slope, in which the corresponding equations of motion depend on both traction coefficients. Therefore, a pair $(\eta, \tilde{\eta})$ determines the range of compensation of the gravity effects (transverse and longitudinal) during motion on the slippery slope, in particular, the behaviour of the time-optimal trajectories. In consequence, we can state that there exist in fact infinitely many slippery slope problems, where the classic Matsumoto's slope-of-a-mountain and Zermelo's navigation under gravitational wind stand for natural but also very particular cases now, among many others.

Furthermore, it will be possible to create the direct links between two arbitrary problems via the general solution, i.e. the transitions $\mathcal{T}_{\mathcal{P}}^{\mathcal{P}'}$, where the traction coefficients are not fixed but varying as in SLIPPERY and S-CROSS thus far. In other words, one can set up the ranges of the parameters, i.e. $\eta \in [\eta_1, \eta_1'] \subseteq [0,1]$, $\tilde{\eta} \in [\tilde{\eta}_1, \tilde{\eta}_1'] \subseteq [0,1]$ and fix the relation $\tilde{\eta} = f(\eta)$, e.g. $\tilde{\eta} = (\eta - \eta_1)(\tilde{\eta}_1' - \tilde{\eta}_1)/(\eta_1' - \eta_1) + \tilde{\eta}_1$. This is visualized graphically by a straight line segment (black) along which $\mathcal{P}$ is moving smoothly, connecting two specific problems $\mathcal{P}_1 = (\eta_1, \tilde{\eta}_1)$, $\mathcal{P}_1' = (\eta_1', \tilde{\eta}_1') \in \tilde{\mathcal{S}}$ in Figure 9.1.

Taking into account (9.4), (9.6) and (9.8), the general equation of motion is formulated as follows

$$v_{\eta\tilde{\eta}} = u + \mathbf{G}_{\eta\tilde{\eta}}, \tag{9.9}$$

[6] Unlike R-ZNP, where $\tilde{\eta} = \eta$.

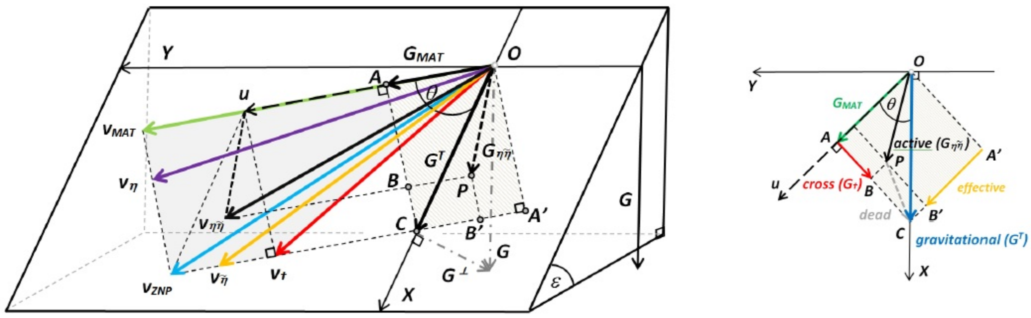


Figure 9.2: Left: A model of a slippery $(\eta, \tilde{\eta})$-slope as an inclined plane M of the slope angle ε in $\mathbb{R}^3$, under the gravity field $\mathbf{G} = \mathbf{G}^T + \mathbf{G}^{\perp}$, acting perpendicularly to the horizontal plane (the base of the slope). The gravitational wind $\mathbf{G}^T$ "blows" tangentially to M in the steepest downhill direction X, and $\mathbf{G}^{\perp}$ is the component of gravity normal to the slope M; $OX \perp OY$, $O \in M$. The resultant velocity is represented by $v_{\eta\tilde{\eta}}$ (black) and its particular (cornered) cases are: v_{ZNP} (blue), v_{MAT} (green), $v_{\dagger}$ (CROSS, red) and $v_{RIEM} = u$ (dashed black). For comparison, v_{η} (purple) and $v_{\tilde{\eta}}$ (yellow) refer to the exemplary SLIPPERY and S-CROSS, respectively. Right: The decompositions of a gravitational wind $\mathbf{G}^T = \overrightarrow{OC}$ (blue) on the slippery slope M, where $OA \perp OA'$. $\mathbf{G}^T$ is a vector sum of an active wind $\mathbf{G}_{\eta\tilde{\eta}} = \overrightarrow{OP}$ (black) and a dead wind $\overrightarrow{PC}$ (dashed grey). The lateral ($\mathbf{G}_{\dagger} = \overrightarrow{AB}$, a cross wind, red) and longitudinal ($\overrightarrow{A'B'}$, an effective wind, yellow) components of the active wind $\mathbf{G}_{\eta\tilde{\eta}} = \overrightarrow{OP}$ w.r.t. u depend in particular on the cross-traction (η) and along-traction ($\tilde{\eta}$) coefficients, respectively, where $\overrightarrow{AB} \perp \overrightarrow{A'B'}$. The direction θ of motion not influenced by gravity and indicated by the Riemannian self-velocity u (the control vector, dashed black) is measured clockwise from OX, where $||u||_h = 1$.

where active wind $\mathbf{G}_{\eta\tilde{\eta}}$ on the $(\eta, \tilde{\eta})$-slope is defined by the following linear combination

$$\mathbf{G}_{\eta\tilde{\eta}} = (1-\eta)\mathbf{G}^{\perp}_{MAT} + (1-\tilde{\eta})\mathbf{G}_{MAT}, \quad (\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \tag{9.10}$$

which is equivalent to

$$\mathbf{G}_{\eta\tilde{\eta}} = (\eta - \tilde{\eta})\mathbf{G}_{MAT} + (1-\eta)\mathbf{G}^T. \tag{9.11}$$

The formula (9.9) implies in particular the equations of motion from the preceding studies of time-optimal navigation on the mountain slopes under gravity, namely:

- $v_{\eta} = u + \mathbf{G}_{\eta} = u + \eta\mathbf{G}_{MAT} + (1-\eta)\mathbf{G}^T$ as in [10] (SLIPPERY), Chapter 7;
- $v_{\dagger} = v_{01} = u + \mathbf{G}_{\dagger} = u - \mathbf{G}_{MAT} + \mathbf{G}^T$ as in [11] (CROSS), Chapter 8;
- $v_{\tilde{\eta}} = u + \mathbf{G}_{\tilde{\eta}} = u - \tilde{\eta}\mathbf{G}_{MAT} + \mathbf{G}^T$ as in [12] (S-CROSS), Section 8.3;
- $v_{MAT} = v_{10} = u + \mathbf{G}_{MAT}$ as in [106] (MAT),
- $v_{ZNP} = v_{00} = u + \mathbf{G}^T$ as in [71, 45] (ZNP) and obviously,
- $v_{11} = u$ (RIEM).

Moreover, for instance, the relation $v_{\eta\tilde{\eta}} = u + \mathbf{G}_{\eta\tilde{\eta}}$, where

$$\mathbf{G}_{\eta\tilde{\eta}} = (1-\eta)\mathbf{G}^{\perp}_{MAT} + \eta\mathbf{G}_{MAT} = (2\eta - 1)\mathbf{G}_{MAT} + (1-\eta)\mathbf{G}^T$$

linking MAT and CROSS has not been covered so far. However, now the corresponding time-minimizing solution will come as the particular case of the general result presented in the sequel.

9.3 Proofs of the main results

The goal of this section is to prove Theorems 9.1.1 and 9.1.2. Some preparations are necessary. Let us consider the n-dimensional Riemannian manifold (M, h), $n > 1$, which represents here a model for a slippery slope of a mountain. Let $\mathbf{G}^T = -\bar{g}\omega^\sharp = -\bar{g}h^{ji}\frac{\partial p}{\partial x^j}\frac{\partial}{\partial x^i}$ be the gravitational wind and let u be the self-velocity of a moving craft on the slope, assuming throughout this section that $||u||_h = 1$. For now, let us consider the active wind $\mathbf{G}_{\eta\tilde{\eta}}$ expressed by (9.10), with $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}} = [0, 1] \times [0, 1]$, which vanishes only when $\eta = \tilde{\eta} = 1$.

9.3.1 $(\eta, \tilde{\eta})$-slope metric

We pose the navigation problem $\mathcal{P}_{\eta,\tilde{\eta}}$ on (M, h) under the action of the active wind $\mathbf{G}_{\eta\tilde{\eta}}$, where the resultant velocity is $v_{\eta\tilde{\eta}} = u + \mathbf{G}_{\eta\tilde{\eta}}$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}}$. Apparently, it looks like a standard Zermelo navigation problem, where the solution is given by a Finsler metric of Randers type if the wind is weak [45, 71, 154, 61]. In reality, our navigation problem is quite complicated due to the active wind $\mathbf{G}_{\eta\tilde{\eta}}$. The key observation is that our ingredient $\mathbf{G}_{\eta\tilde{\eta}}$, written as $\mathbf{G}_{\eta\tilde{\eta}} = (\eta - \tilde{\eta})\mathbf{G}_{MAT} + (1 - \eta)\mathbf{G}^T$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}}$, is not a priori known because only the gravitational wind $\mathbf{G}^T$ is given, and the vector $\mathbf{G}_{MAT}$ being the orthogonal projection of $\mathbf{G}^T$ onto the self-velocity u, depends on the direction of u. Moreover, using (9.10), it follows immediately that $||\mathbf{G}_{\eta\tilde{\eta}}||_h \leq ||\mathbf{G}^T||_h$, for any $(\eta, \tilde{\eta}) \in \mathcal{S}$, where $\mathcal{S} = \tilde{\mathcal{S}} \setminus \{(1, 1)\}$.

As in Chapters 7 and 8 or [10, 12], it is appropriate for us to split the proof of Theorem 9.1.1 into two steps including a sequence of cases and lemmas, which enable us to describe the $(\eta, \tilde{\eta})$-slope metrics besides the necessary and sufficient conditions for their strong convexity, expressed exclusively with respect to the force of the gravitational wind $\mathbf{G}^T$, for any $(\eta, \tilde{\eta}) \in \mathcal{S}$.

The first step describes a direction-dependent deformation, more precisely, the deformation of the background Riemannian metric h by the vector field $(\eta - \tilde{\eta})\mathbf{G}_{MAT}$. The second step develops the classic Zermelo navigation, where the indicatrix of the resulting Finsler metric F of Matsumoto type, provided by the first step, is rigidly translated by the gravitational vector field $(1 - \eta)\mathbf{G}^T$, under the condition $F(x, -(1 - \eta)\mathbf{G}^T) < 1$ which practically secures that a craft on the slippery mountainside can go forward in any direction (see [127, 61]).

Step I. We state that under the assumption that $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h < 1$, the direction-dependent deformation of the Riemannian metric h by $(\eta - \tilde{\eta})\mathbf{G}_{MAT}$ leads to a Finsler metric if and only if $||\mathbf{G}^T||_h < \frac{1}{2|\eta - \tilde{\eta}|}$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$, where $\mathcal{L} = \left\{(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \mid \eta = \tilde{\eta}\right\}$. Moreover, when $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h \geq 1$ at some directions, this deformation cannot afford a Finsler metric.

To this end, we describe the deformation of h by the vector field $(\eta - \tilde{\eta})\mathbf{G}_{MAT}$ in terms of the resultant velocity $v = u + (\eta - \tilde{\eta})\mathbf{G}_{MAT}$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$. Evidently, if $\eta = \tilde{\eta}$, then $v = u$. Furthermore, we need to study the cases:

1. $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h < 1$, 2. $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h = 1$ and 3. $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h > 1$,

separately.

Let us fix a few notations. The desired direction of own motion, denoted by θ, is the angle between $\mathbf{G}^T$ and u. Since $\mathbf{G}_{MAT}$ =$\text{Proj}_u\mathbf{G}^T$, the vectors v, u and $\mathbf{G}_{MAT}$ are collinear and once we have denoted by $\bar{\theta}$ the angle between u and $\mathbf{G}_{MAT}$, it results that it can only be 0 or π. We notice that when θ is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, the angle $\bar{\theta}$ is not determined, i.e. u and $\mathbf{G}^T$ are orthogonal, and $\mathbf{G}_{MAT}$ vanishes.

Case 1. $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h<1$. Since we have assumed that $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h<1$, it is certainly true that the angle between $\mathbf{G}^T$ and v is also θ (the vectors u and v point in the same direction). It is worthwhile to emphasize that due to the assertion $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h<1$ there is not any direction where the resultant vector v vanishes. Now we focus on $\bar{\theta}$, namely:

i) When $\bar{\theta}=0$ (going downhill), we have $\theta\in[0,\frac{\pi}{2})\cup(\frac{3\pi}{2},2\pi)$ and the angle between $\mathbf{G}^T$ and $\mathbf{G}_{MAT}$ is either θ or $2\pi-\theta$. Then, we clearly obtain that $||\mathbf{G}_{MAT}||_h=||\mathbf{G}^T||_h\cos\theta$ and

$$h(v,\mathbf{G}_{MAT})=||v||_h||\mathbf{G}_{MAT}||_h=||v||_h||\mathbf{G}^T||_h\cos\theta=h(v,\mathbf{G}^T).$$

Also, for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}\smallsetminus\mathcal{L}$ it follows that $|\eta-\tilde{\eta}|\ ||\mathbf{G}^T||_h\cos\theta<1$ and $\frac{h(v,\mathbf{G}^T)}{||v||_h}<\frac{1}{|\eta-\tilde{\eta}|}$.

ii) When $\bar{\theta}=\pi$ (going uphill), it turns out that $\theta\in(\frac{\pi}{2},\frac{3\pi}{2})$ and the angle between $\mathbf{G}^T$ and $\mathbf{G}_{MAT}$ is $|\pi-\theta|$. Thus, one easily obtains that $||\mathbf{G}_{MAT}||_h=-||\mathbf{G}^T||_h\cos\theta$ and

$$h(v,\mathbf{G}_{MAT})=-||v||_h||\mathbf{G}_{MAT}||_h=||v||_h||\mathbf{G}^T||_h\cos\theta=h(v,\mathbf{G}^T).$$

Moreover, for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}\smallsetminus\mathcal{L}$ we have $-|\eta-\tilde{\eta}|\ ||\mathbf{G}^T||_h\cos\theta<1$ and $-\frac{h(v,\mathbf{G}^T)}{||v||_h}<\frac{1}{|\eta-\tilde{\eta}|}$.

To sum up, by both of the above sub-cases and noting that $v=u$ when $\theta\in\{\frac{\pi}{2},\frac{3\pi}{2}\}$, we get

$$h(v,\mathbf{G}_{MAT})=h(v,\mathbf{G}^T)=||v||_h||\mathbf{G}^T||_h\cos\theta,\ \text{for any}\ \theta\in[0,2\pi). \tag{9.12}$$

In addition, we can write the inequality $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h<1$ as follows

$$|\eta-\tilde{\eta}|\ ||\mathbf{G}^T||_h|\cos\theta|<1 \qquad \text{or} \qquad \frac{|h(v,\mathbf{G}^T)|}{||v||_h}<\frac{1}{|\eta-\tilde{\eta}|}, \tag{9.13}$$

for any $\theta\in[0,2\pi)$ and $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}\smallsetminus\mathcal{L}$. Now, using (9.12), we proceed by straightforward computation starting with $1=||u||_h=||v-(\eta-\tilde{\eta})\mathbf{G}_{MAT}||_h$. This leads to the equation

$$||v||_h^2-2(\eta-\tilde{\eta})||v||_h||\mathbf{G}^T||_h\cos\theta-[1-(\eta-\tilde{\eta})^2||\mathbf{G}^T||_h^2\cos^2\theta]=0,$$

which, due to the first inequality in (9.13), admits the unique positive root

$$||v||_h=1+(\eta-\tilde{\eta})||\mathbf{G}^T||_h\cos\theta, \tag{9.14}$$

for any $\theta\in[0,2\pi)$ and $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}\smallsetminus\mathcal{L}$.

If we introduce the notation $g_1(x,v)=||v||_h^2-||v||_h-(\eta-\tilde{\eta})h(v,\mathbf{G}^T)$ and we use (9.12), the equation (9.14) can be written into its equivalent form $g_1(x,v)=0$. Thus, based on Okubo's method [106], we can get the function

$$F(x,v)=\frac{||v||_h^2}{||v||_h+(\eta-\tilde{\eta})h(v,\mathbf{G}^T)}$$

as the solution of the equation $g_1(x, \frac{v}{F}) = 0$. Moreover, we can extend $F(x, v)$ to an arbitrary nonzero vector $y \in T_xM$, for any $x \in M$ because any nonzero y can be expressed as $y = cv$, $c > 0$, and $F(x, v) = 1$. Namely, it turns out the following positive homogeneous C^∞-function on TM_0

$$F(x,y) = \frac{||y||_h^2}{||y||_h + (\eta - \tilde{\eta})h(y, \mathbf{G}^T)}\ , \quad \text{for any} \quad (\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}. \tag{9.15}$$

There is still a certain amount of properties which is arising regarding function $F(x, y)$ obtained in (9.15). Before all else, we claim that our assertion $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h < 1$ is a necessary and sufficient condition for $F(x, y)$ to be positive on all TM_0. In order to prove this let us observe that the positivity of (9.15) on TM_0 means that

$$||y||_h + (\eta - \tilde{\eta})h(y, \mathbf{G}^T) > 0, \tag{9.16}$$

for all nonzero y and any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$. If the positivity is achieved on TM_0, we can replace y with $\pm\mathbf{G}^T \neq 0$ in (9.16) and thus, it follows that $|\eta - \tilde{\eta}|\ ||\mathbf{G}^T||_h < 1$. Having the inequality $||\mathbf{G}_{MAT}||_h \leq ||\mathbf{G}^T||_h$ in any direction (note that $||\mathbf{G}_{MAT}||_h = ||\mathbf{G}^T||_h|\cos\theta|$, for any $\theta \in [0, 2\pi)$), it yields that $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h < 1$ on all TM_0. Conversely, if $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h < 1$ on all TM_0 (the posibility that $\mathbf{G}_{MAT} = 0$ is also included), we have $\frac{|h(y,\mathbf{G}^T)|}{||y||_h} < \frac{1}{|\eta - \tilde{\eta}|}$ for any nonzero y, which gives (9.16). Thus, the claim that $F(x, y)$ is positive on TM_0 is proved.

From now on, we use the notations as in Chapters 7 and 8 or [10, 11, 12], that is

$$\alpha^2 = ||y||_h^2 = h_{ij}y^iy^j \qquad \text{and} \quad \beta = -\frac{1}{\bar{g}}h(y, \mathbf{G}^T) = h(y, \omega^\sharp) = b_iy^i, \tag{9.17}$$

$\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ and $||\beta||_h = ||\omega^\sharp||_h$. We notice that the differential 1-form β is closed, i.e. $s_{ij} = 0$, because it includes the gravitational wind $\mathbf{G}^T$ which is a scaled gradient vector field, i.e. $\mathbf{G}^T = -\bar{g}\omega^\sharp = -\bar{g}h^{ji}\frac{\partial p}{\partial x^j}\frac{\partial}{\partial x^i}$; for more details, see [10, Lemma 4.3].

With the notations (9.17), we can express the function (9.15) as

$$F(x,y) = \frac{\alpha^2}{\alpha - (\eta - \tilde{\eta})\bar{g}\beta}, \ \text{for any} \quad (\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}, \tag{9.18}$$

which shows that it is of Matsumoto type having the explicit indicatrix

$$I_F = \big\{(x,y) \in TM_0 \mid \alpha^2[\alpha - (\eta - \tilde{\eta})\bar{g}\beta]^{-1} = 1\big\} \subset TM.$$

Since $y = 0$ does not lie in the closure of the indicatrix I_F, we can extend $F(x, y)$ continuously to all TM, i.e. $F(x, 0) = 0$ for any $x \in M$ (see [61]).

The function (9.18) seems to be a promising Finsler metric. In order to make sure of this, we are going to establish the necessary and sufficient conditions for the strong convexity of the indicatrix I_F, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$. We can write $F(x, y) = \alpha\phi(s)$, where $\phi(s) = \frac{1}{1-(\eta-\tilde{\eta})\bar{g}s}$ with $s = \frac{\beta}{\alpha}$, and the second inequality in (9.13) is actually $|s| < \frac{1}{|\eta-\tilde{\eta}|\bar{g}}$, for arbitrary nonzero $y \in T_xM$ and $x \in M$. Thus, for every $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$ it follows that ϕ is a positive C^∞-function on the open interval $\mathcal{I} = \big(-(|\eta - \tilde{\eta}|\bar{g})^{-1}, (|\eta - \tilde{\eta}|\bar{g})^{-1}\big)$.

In the sequel, we collect the desired properties for $\phi(s)$, with $|s| < \frac{1}{|\eta-\tilde{\eta}|\bar{g}}$, and we control force of the gravitational wind $\mathbf{G}^T$ via the variable s.

Lemma 9.3.1. *Let ϕ be the function given by $\phi(s) = \frac{1}{1-(\eta-\tilde{\eta})\bar{g}s}$ with $s \in \mathcal{I}$. For any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$, the following statements are equivalent:*
i) $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, where $b = ||\omega^\sharp||_h$;
ii) $|s| \leq b < b_0$, where $b_0 = \frac{1}{2|\eta-\tilde{\eta}|\bar{g}}$;
iii) $||\mathbf{G}^T||_h < \frac{1}{2|\eta-\tilde{\eta}|}$.

Proof. By using the Cauchy-Schwarz inequality $|h(y, \omega^\sharp)| \leq ||y||_h||\omega^\sharp||_h$ it follows that $|s| \leq ||\omega^\sharp||_h = b$, for any nonzero $y \in T_xM$ and $x \in M$. Let us now focus on $(b^2 - s^2)\phi''(s)$. Due to $|s| < \frac{1}{|\eta-\tilde{\eta}|\bar{g}}$, one has $(b^2 - s^2)\phi''(s) = (b^2 - s^2)\frac{2(\eta-\tilde{\eta})^2\bar{g}^2}{[1-(\eta-\tilde{\eta})\bar{g}s]^3} \geq 0$. Thus, the minimum value of $(b^2 - s^2)\phi''(s)$ is 0 and it is achieved when $|s| = b$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$. Moreover, a simple computation leads to

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s) = \frac{[1-(\eta-\tilde{\eta})\bar{g}s][1-2(\eta-\tilde{\eta})\bar{g}s]+2(b^2-s^2)(\eta-\tilde{\eta})^2\bar{g}^2}{[1-(\eta-\tilde{\eta})\bar{g}s]^3}. \quad (9.19)$$

To prove i) $\Rightarrow$ ii), we assume that $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$. Let us take $s = \pm b$ in (9.19). It follows that $1 \mp 2(\eta - \tilde{\eta})\bar{g}b > 0$, and then $b < \frac{1}{2|\eta-\tilde{\eta}|\bar{g}}$. Therefore, $|s| \leq b < \frac{1}{2|\eta-\tilde{\eta}|\bar{g}}$ which is precisely the required ii). Conversely, let us assume that $|s| \leq b < \frac{1}{2|\eta-\tilde{\eta}|\bar{g}}$. Due to (9.19), we get

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) \geq \frac{[1-(\eta-\tilde{\eta})\bar{g}s][1-2(\eta-\tilde{\eta})\bar{g}s]}{[1-(\eta-\tilde{\eta})\bar{g}s]^3} = \frac{1-2(\eta-\tilde{\eta})\bar{g}s}{[1-(\eta-\tilde{\eta})\bar{g}s]^2} > 0,$$

for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$.
Now, we prove the implication iii) $\Rightarrow$ ii). Since $||\mathbf{G}^T||_h < \frac{1}{2|\eta-\tilde{\eta}|}$ and $|s| \leq ||\omega^\sharp||_h = b = \frac{1}{\bar{g}}||\mathbf{G}^T||_h$, it turns out $|s| \leq b < \frac{1}{2|\eta-\tilde{\eta}|\bar{g}}$. The implication ii) $\Rightarrow$ iii) is trivial. □

It is worth mentioning that the statement $|s| \leq b < \frac{1}{2|\eta-\tilde{\eta}|\bar{g}}$ also implies that for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$, $\phi(s) - s\phi'(s) > 0$. By the above findings and applying [71, Lemma 1.1.2] and Proposition 6.2.1, we have stated the following result.

Lemma 9.3.2. *For any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$, $F(x, y) = \frac{\alpha^2}{\alpha-(\eta-\tilde{\eta})\bar{g}\beta}$ is a Finsler metric if and only if $||\mathbf{G}^T||_h < \frac{1}{2|\eta-\tilde{\eta}|}$.*

Therefore, once we have Lemma 9.3.2, we conclude that the indicatrix I_F is strongly convex if and only if $||\mathbf{G}^T||_h < \frac{1}{2|\eta-\tilde{\eta}|}$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$.

Case 2. $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h = 1$. We start by assuming that $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h = 1$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$. Observe that a traverse of the mountain, i.e. when $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, cannot be followed here. Indeed, when $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, $\mathbf{G}_{MAT}$ vanishes which contradicts our assumption. Moreover, since $||\mathbf{G}_{MAT}||_h \leq ||\mathbf{G}^T||_h$, it follows that $||\mathbf{G}^T||_h \geq \frac{1}{|\eta-\tilde{\eta}|}$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$. In the sequel, we have to analyze the aforementioned possibilities for $\bar{\theta}$:

i) when $\bar{\theta} = 0$, we clearly have $\theta \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$ and $||\mathbf{G}_{MAT}||_h = ||\mathbf{G}^T||_h \cos\theta$. In addition, our assumption implies that $u = |\eta - \tilde{\eta}|\mathbf{G}_{MAT}$, and thus

$$v = (\eta - \tilde{\eta} + |\eta - \tilde{\eta}|)\mathbf{G}_{MAT} = \begin{cases} 2(\eta - \tilde{\eta})\mathbf{G}_{MAT}, & \text{if } \eta > \tilde{\eta} \\ 0, & \text{if } \eta < \tilde{\eta} \end{cases}.$$

Based on this, we next get

(a) if $\eta > \tilde{\eta}$, then $||v||_h = 2(\eta - \tilde{\eta})||\mathbf{G}_{MAT}||_h = 2$ and

$$h(v, \mathbf{G}_{MAT}) = ||v||_h||\mathbf{G}_{MAT}||_h = ||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T);$$

(b) if $\eta < \tilde{\eta}$, the resultant velocity v vanishes, while attempting to go down the slope.

ii) when $\bar{\theta} = \pi$, then $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and thus $u = -|\eta - \tilde{\eta}|\mathbf{G}_{MAT}$ and $||\mathbf{G}_{MAT}||_h = -||\mathbf{G}^T||_h \cos\theta$. It turns out that

$$v = (\eta - \tilde{\eta} - |\eta - \tilde{\eta}|)\mathbf{G}_{MAT} = \begin{cases} 0, & \text{if } \eta > \tilde{\eta} \\ 2(\eta - \tilde{\eta})\mathbf{G}_{MAT}, & \text{if } \eta < \tilde{\eta} \end{cases}.$$

Also, this must be splitted into

(a) if $\eta > \tilde{\eta}$, the resultant velocity v vanishes, while attempting to go up the slope;

(b) if $\eta < \tilde{\eta}$, then $||v||_h = -2(\eta - \tilde{\eta})||\mathbf{G}_{MAT}||_h = 2$ and

$$h(v, \mathbf{G}_{MAT}) = -||v||_h||\mathbf{G}_{MAT}||_h = ||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T).$$

Summing up the above findings, when v does not vanish, we have $||v||_h = 2$ and among the directions corresponding to θ, only such directions for which $\cos\theta = \frac{1}{(\eta - \tilde{\eta})\ ||\mathbf{G}^T||_h}$ (i.e. $\frac{h(v, \mathbf{G}^T)}{||v||_h} = \frac{1}{\eta - \tilde{\eta}}$), for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$, can be followed in this case. Let us consider $g_2(x, v) = 0$, where $g_2(x, v) = ||v||_h - 2$. By Okubo's method [106], we get the function

$$F(x, v) = \frac{1}{2}||v||_h, \tag{9.20}$$

as the solution of the equation $g_2(x, \frac{v}{F}) = 0$.

The extension of $F(x, v)$ to an arbitrary nonzero vector $y \in \mathcal{A}_x = \mathcal{A} \cap T_xM$, for any $x \in M$, is $F(x, y) = \frac{1}{2}||y||_h$, where $\mathcal{A} = \{(x, y) \in TM_0 \mid ||y||_h - (\eta - \tilde{\eta})h(y, \mathbf{G}^T) = 0\}$. Since $||\mathbf{G}^T||_h \geq \frac{1}{|\eta - \tilde{\eta}|}$ it follows that $\mathbf{G}^T \in \mathcal{A}_x$ if and only if $\mathbf{G}^T = \mathbf{G}_{MAT}$ and $\eta > \tilde{\eta}$ (here the angle θ can only be 0) and $-\mathbf{G}^T \in \mathcal{A}_x$ if and only if $\mathbf{G}^T = \mathbf{G}_{MAT}$ and $\eta < \tilde{\eta}$ (here the angle θ can only be π). Anyway, this case does not povide a proper Finsler metric.

Case 3. $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h > 1$. The remaining case is $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h > 1$. On one hand, it implies that $||\mathbf{G}^T||_h > \frac{1}{|\eta - \tilde{\eta}|}$, as well as the fact that θ cannot be $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. Indeed, if $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, then $v = u$ and thus, G_{MAT} vanishes, which contradicts our assumption. On the other hand, it follows that for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$, the resultant velocity v and $(\eta - \tilde{\eta})\mathbf{G}_{MAT}$ point in the same direction (downhill when $\eta > \tilde{\eta}$ and uphill when $\eta < \tilde{\eta}$) and $h(v, \mathbf{G}_{MAT}) = \frac{|\eta - \tilde{\eta}|}{\eta - \tilde{\eta}}||v||_h||\mathbf{G}_{MAT}||_h$. Moreover, $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h > 1$ states that there is not any direction where the resultant vector v vanishes. Again, we have to take into consideration both possibilities for $\bar{\theta}$. Namely,

i) when $\bar{\theta} = 0$, then $\theta \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$. In particular, we have $||\mathbf{G}_{MAT}||_h = ||\mathbf{G}^T||_h \cos\theta$ and due to the required assumption, it follows that $|\eta - \tilde{\eta}|\ ||\mathbf{G}^T||_h \cos\theta > 1$. Two possibilities must still be analyzed:

(a) if $\eta > \tilde{\eta}$, then $\measuredangle(\mathbf{G}^T, v) \in \{\theta, 2\pi - \theta\}$. Thus, we get

$$h(v, \mathbf{G}_{MAT}) = ||v||_h||\mathbf{G}_{MAT}||_h = ||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T)$$

and $\frac{h(v,\mathbf{G}^T)}{||v||_h} > \frac{1}{\eta-\tilde{\eta}}$;

(b) if $\eta < \tilde{\eta}$, then $\measuredangle(\mathbf{G}^T, v) \in \{\pi+\theta, \theta-\pi\}$ as well as

$$h(v, \mathbf{G}_{MAT}) = -||v||_h||\mathbf{G}_{MAT}||_h = -||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T)$$

and $-\frac{h(v,\mathbf{G}^T)}{||v||_h} > \frac{1}{\tilde{\eta}-\eta}$.

ii) when $\bar{\theta} = \pi$, one gets $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$. In particular, we have $||\mathbf{G}_{MAT}||_h = -||\mathbf{G}^T||_h \cos\theta$ and $|\eta-\tilde{\eta}|\ ||\mathbf{G}^T||_h \cos\theta < -1$ since $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h > 1$. Moreover,

(a) if $\eta > \tilde{\eta}$, then $\measuredangle(\mathbf{G}^T, v) = |\theta-\pi|$. Thus,

$$h(v, \mathbf{G}_{MAT}) = ||v||_h||\mathbf{G}_{MAT}||_h = -||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T)$$

and $\frac{h(v,\mathbf{G}^T)}{||v||_h} > \frac{1}{\eta-\tilde{\eta}}$;

(b) if $\eta < \tilde{\eta}$, then the angle between $\mathbf{G}^T$ and v is also θ. In consequence, it turns out that

$$h(v, \mathbf{G}_{MAT}) = -||v||_h||\mathbf{G}_{MAT}||_h = ||v||_h||\mathbf{G}^T||_h \cos\theta = h(v, \mathbf{G}^T)$$

and $-\frac{h(v,\mathbf{G}^T)}{||v||_h} > \frac{1}{\tilde{\eta}-\eta}$.

By combining the above possibilities and since $\mathbf{G}_{MAT}$ cannot be vanished, we get

$$h(v, \mathbf{G}_{MAT}) = h(v, \mathbf{G}^T) = \frac{|\eta-\tilde{\eta}|}{\eta-\tilde{\eta}}||v||_h||\mathbf{G}^T||_h|\cos\theta|, \text{ for any } \theta \in [0, 2\pi) \smallsetminus \{\pi/2, 3\pi/2\}, \tag{9.21}$$

and the condition $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h > 1$ is equivalent to

$$|\cos\theta| > \frac{1}{|\eta-\tilde{\eta}|\ ||\mathbf{G}^T||_h} \quad \text{or} \quad \begin{cases} \frac{h(v,\mathbf{G}^T)}{||v||_h} > \frac{1}{\eta-\tilde{\eta}}, & \text{if } \eta > \tilde{\eta} \\ -\frac{h(v,\mathbf{G}^T)}{||v||_h} > \frac{1}{\tilde{\eta}-\eta}, & \text{if } \eta < \tilde{\eta} \end{cases}, \quad \text{or} \quad (\eta-\tilde{\eta})\frac{h(v,\mathbf{G}^T)}{||v||_h} > 1, \tag{9.22}$$

for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \smallsetminus \mathcal{L}$.

Therefore, among the directions corresponding to $\theta \in [0, 2\pi) \smallsetminus \{\pi/2, 3\pi/2\}$ only such directions for which $|\cos\theta| > \frac{1}{|\eta-\tilde{\eta}|\ ||\mathbf{G}^T||_h}$ can be followed in this case. By using (9.21), a simple computation, starting with $1 = ||u||_h = ||v-(\eta-\tilde{\eta})\mathbf{G}_{MAT}||_h$, leads to the equation

$$||v||_h^2 - 2|\eta-\tilde{\eta}|\ ||v||_h||\mathbf{G}^T||_h \cos\theta - [1-(\eta-\tilde{\eta})^2||\mathbf{G}^T||_h^2\cos^2\theta] = 0.$$

Since $|\cos\theta| > \frac{1}{|\eta-\tilde{\eta}|\ ||\mathbf{G}^T||_h}$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \smallsetminus \mathcal{L}$ and $\theta \in [0, 2\pi) \smallsetminus \{\pi/2, 3\pi/2\}$, it follows that the last equation admits two positive roots

$$||v||_h = \pm 1 + |\eta-\tilde{\eta}|\ ||\mathbf{G}^T||_h|\cos\theta|. \tag{9.23}$$

Based on the property (9.21), we can write (9.23) as $g_3(x, v) = 0$, where

$$g_3(x, v) = ||v||_h^2 \mp ||v||_h - (\eta-\tilde{\eta})h(v, \mathbf{G}^T).$$

If we apply Okubo's method again [106], we get the following positive functions $F_{1,2}(x, v) = \frac{||v||_h^2}{\pm||v||_h + (\eta-\tilde{\eta})h(v,\mathbf{G}^T)}$ as the solutions of the equation $g_3(x, \frac{v}{F}) = 0$. Next, we extend $F_{1,2}(x, v)$ to an arbitrary nonzero vector $y \in \mathcal{A}_x^* = \mathcal{A}^* \cap T_xM$, for any $x \in M$, where

$$\mathcal{A}^* = \{(x, y) \in TM \mid\ ||y||_h - (\eta-\tilde{\eta})h(y, \mathbf{G}^T) < 0\} \tag{9.24}$$

is an open conic subset of TM_0, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$. Thus, we obtain the positive homogeneous C^{∞}-functions

$$F_{1,2}(x, y) = \frac{||y||_h^2}{\pm||y||_h + (\eta - \tilde{\eta})h(y, \mathbf{G}^T)} \tag{9.25}$$

on $\mathcal{A}^*$, with $F_{1,2}(x, v) = 1$. We notice that $\mathbf{G}^T \in \mathcal{A}_x^*$ iff $\eta > \tilde{\eta}$ and $-\mathbf{G}^T \in \mathcal{A}_x^*$ iff $\eta < \tilde{\eta}$. Moreover, due to (9.22), our initial assumption $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h > 1$ is a necessary and sufficient condition for $F_{1,2}(x, y)$ to be positive on $\mathcal{A}^*$.

By the notations (9.17), the functions $F_{1,2}(x, y)$ are of Matsumoto type, namely

$$F_{1,2}(x, y) = \frac{\alpha^2}{\pm\alpha - (\eta - \tilde{\eta})\bar{g}\beta} \tag{9.26}$$

on conic domain $\mathcal{A}^*$, rewritten as $\mathcal{A}^* = \{(x, y) \in TM \mid \alpha + (\eta - \tilde{\eta})\bar{g}\beta < 0\}$. However, $F_{1,2}$ can be at most conic Finsler metrics. By applying [87, Corollary 4.15], it turns out that both $F_{1,2}$ are strongly convex on $\mathcal{A}^*$ and thus, they are conic Finsler metrics on $\mathcal{A}^*$, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$. Indeed, for $F_{1,2}$ the strongly convex conditions $[\alpha \mp 2(\eta - \tilde{\eta})\bar{g}\beta][\alpha \mp (\eta - \tilde{\eta})\bar{g}\beta] > 0$ are satisfied for any $(x, y) \in \mathcal{A}^*$ and $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$.

Summarizing the results from this step, beyond their intrinsic interest, it turns out that the direction-dependent deformation of the background Riemannian metric h by the vector field $(\eta - \tilde{\eta})\mathbf{G}_{MAT}$, with $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h < 1$ performed for every direction (note that the converse inequality $|\eta - \tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h \geq 1$ one may carry out only at some directions), for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$ provides the Finsler metric $F(x, y) = \frac{\alpha^2}{\alpha - (\eta - \tilde{\eta})\bar{g}\beta}$ if and only if $||\mathbf{G}^T||_h < \frac{1}{2|\eta - \tilde{\eta}|}$.

Step II. In attempting to use Proposition 6.1.1, we consider the following navigation data $(F, (1 - \eta)\mathbf{G}^T)$ on the Finsler manifold (M, F), where F is either the Finsler metric (9.18) if $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$ or the background Riemannian metric h if $(\eta, \tilde{\eta}) \in \mathcal{L}$, assuming that

$$F(x, -(1 - \eta)\mathbf{G}^T) < 1. \tag{9.27}$$

Exploring the Zermelo navigation on (M, F) with the aforementioned navigation data $(F, (1 - \eta)\mathbf{G}^T)$, we supply new Finsler metrics, which we call the $(\eta, \tilde{\eta})$-slope metrics, together with the necessary and sufficient conditions for the strong convexity of their indicatrices. More precisely, applying Proposition 6.1.1, for each $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}}$, the $(\eta, \tilde{\eta})$-slope metric has to arise as the unique positive solution $\tilde{F}$ of the equation

$$F(x, y - (1 - \eta)\tilde{F}(x, y)\mathbf{G}^T) = \tilde{F}(x, y), \tag{9.28}$$

for any $(x, y) \in TM_0$.

Before doing this, a few details must be outlined. On one hand, the meaning of our second step is that the addition of the scaled gravitational wind $(1 - \eta)\mathbf{G}^T$ generates a rigid translation to the strongly convex indicatrix provided by $v = u - (\eta - \tilde{\eta})\mathbf{G}_{MAT}$ in the first step, for any $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$ (see [61]). We have already got rid of the possibility that $|\eta - \tilde{\eta}|||\mathbf{G}_{MAT}||_h \geq 1$ since it only supplied conic Finsler metrics and thus, going forward in any direction is not enabled. On the other hand, the condition (9.27) plays an essential role for the resulting indicatrix, obtained by translation, to be strongly convex and to determine a new Finsler metric as the unique solution of (9.28) (see [61, p. 10 and Proposition 2.14]). In other words, the condition (9.27) secures that for any $x \in M$, $y = 0$ belongs to the region bounded by the new translated indicatrix $I_{\tilde{F}}$. Moreover, some additional computations can show that if we replace the conic Finsler metrics from Cases 2 or 3 in the inequality (9.27), it may exist only for some $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}\setminus\mathcal{L}$, which is contrary to what we aimed for from the beginning in order to cover the whole square $\tilde{\mathcal{S}}$.

In the sequel, we expand the left-hand side of (9.28). Let us observe that the Finsler metric F can be written as $F(x,y)=\frac{\alpha^2}{\alpha-(\eta-\tilde{\eta})\bar{g}\beta}$, for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$. In particular, if $\eta=1$, it is obvious that $\tilde{F}(x,y)=\frac{\alpha^2}{\alpha-(1-\tilde{\eta})\bar{g}\beta}$, for any $\tilde{\eta}\in[0,1]$, (i.e. the so-called reduced Matsumoto metric). For arbitrary $\eta\in[0,1]$, by taking $y-(1-\eta)\tilde{F}(x,y)\mathbf{G}^T$ instead of y in (9.17), some standard computations give that

$$\alpha^2\left(x,y-(1-\eta)\tilde{F}(x,y)\mathbf{G}^T\right)=\alpha^2(x,y)+2(1-\eta)\bar{g}\beta(x,y)\tilde{F}(x,y)+(1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}^2(x,y)$$

and

$$\beta\left(x,y-(1-\eta)\tilde{F}(x,y)\mathbf{G}^T\right)=\beta(x,y)+\frac{1-\eta}{\bar{g}}||\mathbf{G}^T||_h^2\tilde{F}(x,y),$$

where we used the relation $\beta(x,\mathbf{G}^T)=-\frac{1}{\bar{g}}||\mathbf{G}^T||_h^2$. Therefore, it turns out that the left-hand side of (9.28) is

$$\frac{\alpha^2+2(1-\eta)\bar{g}\beta\tilde{F}+(1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}^2}{\sqrt{\alpha^2+2(1-\eta)\bar{g}\beta\tilde{F}+(1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}^2}-(\eta-\tilde{\eta})\bar{g}\beta-(\eta-\tilde{\eta})(1-\eta)||\mathbf{G}^T||_h^2\tilde{F}},$$

where α, β and $\tilde{F}$ are evaluated at (x,y). Now, if we substitute this into (9.28), we get the irrational equation

$$\tilde{F}\sqrt{\alpha^2+2(1-\eta)\bar{g}\beta\tilde{F}+(1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}^2}=\alpha^2+(2-\eta-\tilde{\eta})\bar{g}\beta\tilde{F}+(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2\tilde{F}^2, \tag{9.29}$$

which is equivalent to the following polynomial equation

$$\begin{aligned}&(1-\eta)^2||\mathbf{G}^T||_h^2[1-(1-\tilde{\eta})^2\,||\mathbf{G}^T||_h^2]\tilde{F}^4+2(1-\eta)\left[1-(2-\eta-\tilde{\eta})\,(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right]\bar{g}\beta\tilde{F}^3\\ &+\{\left[1-2(1-\eta)\,(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right]\alpha^2-(2-\eta-\tilde{\eta})^2\,\bar{g}^2\beta^2\}\tilde{F}^2-2\,(2-\eta-\tilde{\eta})\,\bar{g}\alpha^2\beta\tilde{F}-\alpha^4=0,\end{aligned} \tag{9.30}$$

for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$.

In the special case where $\eta=\tilde{\eta}=1$, we obviously have $\tilde{F}=h$. Moreover, we note that if $(1-\eta)^2[1-(1-\tilde{\eta})^2\,||\mathbf{G}^T||_h^2]\neq 0$, the last equation admits four roots, and thanks to the condition (9.27), we know precisely that among all roots there is only one positive. For any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$, it should be the $(\eta,\tilde{\eta})$-slope metric.

From now on, we denote by $\tilde{F}_{\eta\tilde{\eta}}$ the $(\eta,\tilde{\eta})$-slope metric, outlining that $\tilde{F}_{\eta\tilde{\eta}}$ satisfies (9.29) and moreover, along any regular piecewise C^∞-curve γ, parametrized by time that represents a trajectory in Zermelo's problem, we have $\tilde{F}_{\eta\tilde{\eta}}(\gamma(t),\dot{\gamma}(t))=1$. This is the time in which a craft or a vehicle goes along γ.

Now, it remains to provide explicitly the necessary and sufficient conditions for the indicatrix of $\tilde{F}_{\eta\tilde{\eta}}$ to be strongly convex, and thus we will outline the argument that the $\tilde{F}_{\eta\tilde{\eta}}$-geodesics locally minimize time. In order to handle this issue, we need to characterize the inequality (9.27) which is equivalent to

$$\frac{1-(1-\tilde{\eta})||\mathbf{G}^T||_h}{1-(\eta-\tilde{\eta})||\mathbf{G}^T||_h}>0. \tag{9.31}$$

Indeed, since $F(x,y)=\frac{\alpha^2}{\alpha-(\eta-\tilde{\eta})\bar{g}\beta}=\frac{||y||_h^2}{||y||_h+(\eta-\tilde{\eta})h(y,\mathbf{G}^T)}$, it turns out that, for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$, the left-hand side of (9.27) is

$$F(x,-(1-\eta)\mathbf{G}^T)=\frac{||-(1-\eta)\mathbf{G}^T||_h^2}{||-(1-\eta)\mathbf{G}^T||_h+(\eta-\tilde{\eta})h(-(1-\eta)\mathbf{G}^T,\mathbf{G}^T)}=\frac{(1-\eta)||\mathbf{G}^T||_h}{1-(\eta-\tilde{\eta})||\mathbf{G}^T||_h}$$

which together with (9.27) conclude the claim (9.31). Namely, we prove the result

Lemma 9.3.3. *The following statements are equivalent:*
i) for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$, the indicatrix $I_{\tilde{F}_{\eta\tilde{\eta}}}$ of the $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$ is strongly convex;
ii) the gravitational wind $\mathbf{G}^T$ is restricted to either $||\mathbf{G}^T||_h<\frac{1}{1-\tilde{\eta}}$ and $(\eta,\tilde{\eta})\in\mathcal{D}_1\cup\mathcal{D}_2$, or $||\mathbf{G}^T||_h<\frac{1}{2|\eta-\tilde{\eta}|}$ and $(\eta,\tilde{\eta})\in\mathcal{D}_3\cup\mathcal{D}_4$, where

$$\begin{aligned}&\mathcal{D}_1=\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\eta\geq\tilde{\eta}>2\eta-1\}, &&\mathcal{D}_2=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\tfrac{3\tilde{\eta}-1}{2}<\eta<\tilde{\eta}\right\},\\ &\mathcal{D}_3=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\eta\geq\tfrac{1}{2},\ \tilde{\eta}\leq2\eta-1\right\}, &&\mathcal{D}_4=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\tilde{\eta}\geq\tfrac{1}{3},\ \eta\leq\tfrac{3\tilde{\eta}-1}{2}\right\},\end{aligned}\tag{9.32}$$

$\mathcal{S}=\bigcup_{i=1}^{4}\mathcal{D}_i$ and $\mathcal{D}_i\cap\mathcal{D}_j=\varnothing$, for any $i\neq j$, $i,j=1,...,4$. No restriction should be imposed on $||\mathbf{G}^T||_h$ if $\eta=\tilde{\eta}=1$.
iii) the active wind $\mathbf{G}_{\eta\tilde{\eta}}$ given by (9.11) is restricted to either $||\mathbf{G}_{\eta\tilde{\eta}}||_h<\frac{1}{1-\tilde{\eta}}$ and $(\eta,\tilde{\eta})\in\mathcal{D}_1\cup\mathcal{D}_2$, or $||\mathbf{G}_{\eta\tilde{\eta}}||_h<\frac{1}{2|\eta-\tilde{\eta}|}$ and $(\eta,\tilde{\eta})\in\mathcal{D}_3\cup\mathcal{D}_4$.

Proof. To prove the equivalence i) $\Leftrightarrow$ ii) one has to take into account (9.31), for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$, (see Figure 9.3). Because of this, the following cases must be analyzed separately:
a) if $\eta>\tilde{\eta}$ and $\eta\neq1$, then the inequality (9.31) yields either $||\mathbf{G}^T||_h<\frac{1}{1-\tilde{\eta}}$ or $||\mathbf{G}^T||_h>\frac{1}{\eta-\tilde{\eta}}$. If we combine these with the strong convexity condition for the indicatrix I_F, more precisely $||\mathbf{G}^T||_h<\frac{1}{2(\eta-\tilde{\eta})}$ for $1>\eta>\tilde{\eta}\geq0$, we obtain:

- $||\mathbf{G}^T||_h<\frac{1}{1-\tilde{\eta}}$ if either $(\eta,\tilde{\eta})\in\mathcal{R}_1$ or $(\eta,\tilde{\eta})\in\mathcal{R}_3$, where

$$\mathcal{R}_1=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid0\leq\tilde{\eta}<\eta<\frac{1}{2}\right\},\ \mathcal{R}_3=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\eta\in\left[\frac{1}{2},1\right),\ \tilde{\eta}\in(2\eta-1,\eta)\right\}.$$

  It is obvious that $\mathcal{R}_1$ and $\mathcal{R}_3$ are subsets of $\mathcal{D}_1$ and $\mathcal{D}_1=\mathcal{R}_1\cup\mathcal{R}_3\cup\mathcal{L}_0$, where $\mathcal{L}_0$ denotes $\mathcal{L}\setminus\{(1,1)\}$. Thus, $||\mathbf{G}^T||_h<\frac{1}{1-\tilde{\eta}}$ if $(\eta,\tilde{\eta})\in\mathcal{R}_1\cup\mathcal{R}_3=\mathcal{D}_1\setminus\mathcal{L}_0$.

- $||\mathbf{G}^T||_h<\frac{1}{2(\eta-\tilde{\eta})}$ if $1>\eta\geq\frac{1}{2}$ and $2\eta-1\geq\tilde{\eta}\geq0$. Hence, we have $||\mathbf{G}^T||_h<\frac{1}{2(\eta-\tilde{\eta})}$ if $(\eta,\tilde{\eta})\in\mathcal{D}_3\setminus\mathcal{L}_1$, where $\mathcal{L}_1=\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\eta=1\}$.

b) if $\eta=\tilde{\eta}$ and $\eta\neq1$, then $F=h$ and the resultant metric is a Randers one in this case. Moreover, for every $(\eta,\tilde{\eta})\in\mathcal{L}_0$, the inequality (9.31) is equivalent to $||\mathbf{G}^T||_h<\frac{1}{1-\eta}$.
c) if $\eta<\tilde{\eta}$ and $\tilde{\eta}\neq1$, then (9.31) implies that $||\mathbf{G}^T||_h<\frac{1}{1-\tilde{\eta}}$. Combining this with the strong convexity condition for the indicatrix I_F, i.e. $||\mathbf{G}^T||_h<\frac{1}{2(\tilde{\eta}-\eta)}$ for $0\leq\eta<\tilde{\eta}<1$, we get:

- $||\mathbf{G}^T||_h < \frac{1}{1-\tilde{\eta}}$ if either $(\eta,\tilde{\eta}) \in \mathcal{R}_2$ or $(\eta,\tilde{\eta}) \in \mathcal{R}_4$, where

$$\mathcal{R}_2 = \left\{(\eta,\tilde{\eta}) \in \mathcal{S} \mid 0 \le \eta < \tilde{\eta} < \frac{1}{3}\right\}, \quad \mathcal{R}_4 = \left\{(\eta,\tilde{\eta}) \in \mathcal{S} \mid \tilde{\eta} \in \left[\frac{1}{3}, 1\right),\ \eta \in \left(\frac{3\tilde{\eta}-1}{2}, \tilde{\eta}\right)\right\}.$$

  It is clear that $\mathcal{R}_2$ and $\mathcal{R}_4$ are subsets of $\mathcal{D}_2$ and $\mathcal{D}_2 = \mathcal{R}_2 \cup \mathcal{R}_4$. Thus, $||\mathbf{G}^T||_h < \frac{1}{1-\tilde{\eta}}$ if $(\eta,\tilde{\eta}) \in \mathcal{D}_2$.

- $||\mathbf{G}^T||_h < \frac{1}{2(\tilde{\eta}-\eta)}$ if $\frac{1}{3} \le \tilde{\eta} < 1$ and $0 \le \eta \le \frac{3\tilde{\eta}-1}{2}$. It follows that $||\mathbf{G}^T||_h < \frac{1}{2(\eta-\tilde{\eta})}$ if $(\eta,\tilde{\eta}) \in \mathcal{D}_4 \setminus \mathcal{L}_2$, where $\mathcal{L}_2 = \{(\eta,\tilde{\eta}) \in \mathcal{S} \mid \tilde{\eta} = 1\}$.

d) if $\eta \ne \tilde{\eta}$ and $\eta = 1$, then $1 > \tilde{\eta}$ and (9.31) is fulfilled. Thus, in this case we have $(1,\tilde{\eta})$-slope metric which is the Matsumoto type metric F ((9.18) with $\eta = 1$), and the strong convexity of its indicatrix yields $||\mathbf{G}^T||_h < \frac{1}{2(1-\tilde{\eta})}$. Consequently, $||\mathbf{G}^T||_h < \frac{1}{2(1-\tilde{\eta})}$ if $(\eta,\tilde{\eta}) \in \mathcal{L}_1$.
e) if $\eta \ne \tilde{\eta}$ and $\tilde{\eta} = 1$, then $\eta < 1$ and the inequality (9.31) holds. It turns out that the strong convexity corresponding to the metric F ((9.18)), certified by Lemma 9.3.2, ensures the strong convexity which corresponds to the $(\eta,1)$-slope metric, namely $||\mathbf{G}^T||_h < \frac{1}{2(1-\eta)}$. So, we have $||\mathbf{G}^T||_h < \frac{1}{2(1-\eta)}$ if $(\eta,\tilde{\eta}) \in \mathcal{L}_2$.
f) if $\eta = \tilde{\eta} = 1$, then there is not any deformation for h since $\mathbf{G}_{\eta\tilde{\eta}} = 0$ and thus, there is no restriction on $||\mathbf{G}^T||_h$.

Summing up the above findings, we obtain that the inequality (9.27) is equivalent to either $||\mathbf{G}^T||_h < \frac{1}{1-\tilde{\eta}}$ and $(\eta,\tilde{\eta}) \in \mathcal{D}_1 \cup \mathcal{D}_2$, or $||\mathbf{G}^T||_h < \frac{1}{2|\eta-\tilde{\eta}|}$ and $(\eta,\tilde{\eta}) \in \mathcal{D}_3 \cup \mathcal{D}_4$.

The argument which proves the equivalence ii) $\Leftrightarrow$ iii) is that $||\mathbf{G}_{\eta\tilde{\eta}}||_h \le ||\mathbf{G}^T||_h$, for any $(\eta,\tilde{\eta}) \in \mathcal{S}$ and, furthermore, the maximum of $||\mathbf{G}_{\eta\tilde{\eta}}||_h$ coincides with $||\mathbf{G}^T||_h$ for $\eta = 0$ (which is possible both in $\mathcal{D}_1 \cup \mathcal{D}_2$ and in $\mathcal{D}_3 \cup \mathcal{D}_4$), since $\mathbf{G}_{MAT}$ must vanish for some directions. □

Based on the results stated in Steps I and II, we have performed the proof of Theorem 9.1.1.

We remark that, according to Lemma 9.3.3, the force of the active wind $\mathbf{G}_{\eta\tilde{\eta}}$ can be accounted for in terms of the force of the gravitational wind $\mathbf{G}^T$, i.e. $||\mathbf{G}^T||_h < \tilde{b}_0$, in the problem $\mathcal{P}_{\eta,\tilde{\eta}}$, for any $(\eta,\tilde{\eta}) \in \mathcal{S}$ (see Figure 9.3) where

$$\tilde{b}_0 = \begin{cases} \frac{1}{1-\tilde{\eta}}, & \text{if } (\eta,\tilde{\eta}) \in \mathcal{D}_1 \cup \mathcal{D}_2 \\ \frac{1}{2|\eta-\tilde{\eta}|}, & \text{if } (\eta,\tilde{\eta}) \in \mathcal{D}_3 \cup \mathcal{D}_4 \end{cases}. \tag{9.33}$$

It is worthwhile to mention a few observations regarding the range of $\tilde{b}_0$. For example, when $(\eta,\tilde{\eta}) \in \mathcal{R}_1$, it follows that $\tilde{b}_0 \in [1,2)$ or when $(\eta,\tilde{\eta}) \in \mathcal{R}_2$, we obtain $\tilde{b}_0 \in (1,\frac{3}{2})$. Moreover, for $(\eta,\tilde{\eta}) \in \mathcal{R}_3 \cup \mathcal{R}_4$, $\tilde{b}_0 \longrightarrow \infty$ as $\tilde{\eta} \nearrow 1$. Similarly, for $(\eta,\tilde{\eta}) \in \mathcal{D}_3 \cup \mathcal{D}_4$, $\tilde{b}_0 \longrightarrow \infty$ as $|\eta-\tilde{\eta}| \longrightarrow 0$. In fact, once we are closer and closer to the point $(0,0) \in \tilde{\mathcal{S}}$, the admitted force of the gravitational wind is weaker because $\tilde{b}_0 \longrightarrow 1$ as $\tilde{\eta} \searrow 0$. On the other hand, the closer we approach to the point $(1,1) \in \tilde{\mathcal{S}}$, the stronger the allowed force of $\mathbf{G}^T$ becomes. However, there is $(\eta,\tilde{\eta}) \in \mathcal{S}$ such that $||\mathbf{G}^T||_h > 1$ and the indicatrix $I_{\tilde{F}_{\eta\tilde{\eta}}}$ of the $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$ is still strongly convex, unlike the classic navigation problems, i.e. ZNP where $||\mathbf{G}^T||_h < 1$ or MAT where $||\mathbf{G}^T||_h < \frac{1}{2}$.

The allowable gravitational wind force $||\mathbf{G}^T||_h < \tilde{b}_0$ for the general slippery slope model determined by the strong convexity conditions, including the influence of both traction coefficients, is illustrated in Figure 9.3, right-hand side.

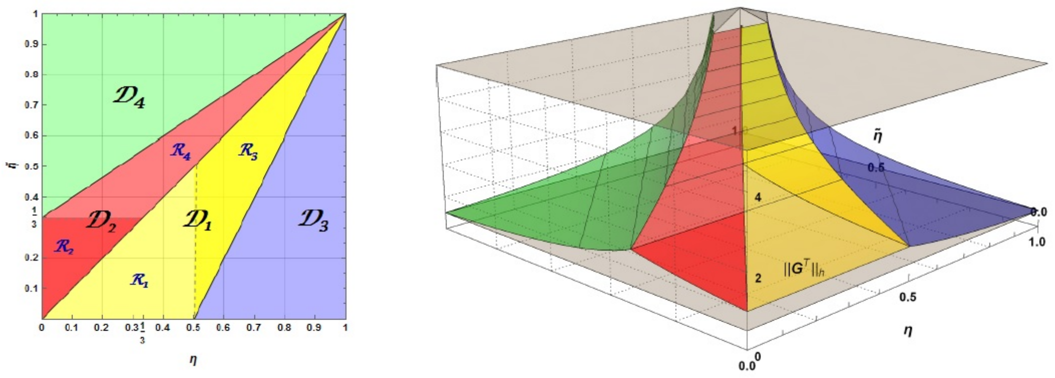


Figure 9.3: Left: The partition of the problem square diagram $\tilde{\mathcal{S}} = \mathcal{S} \cup \{(1,1)\}$ as in (9.32), where $\mathcal{D}_1 = \mathcal{R}_1 \cup \mathcal{R}_3 \cup \mathcal{L}_0$, $\mathcal{D}_2 = \mathcal{R}_2 \cup \mathcal{R}_4$, where $\mathcal{L}_0 = \mathcal{L} \setminus \{(1,1)\}$, and $\mathcal{L} = \left\{(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}} \mid \eta = \tilde{\eta}\right\}$. Right: The allowable gravitational wind force $||\mathbf{G}^T||_h$ in the general slippery slope model determined by the strong convexity conditions given by (9.33), including the influence of both traction coefficients (η, $\tilde{\eta}$), i.e. $||\mathbf{G}^T||_h < \tilde{b}_0$. For clarity of presentation, we limited the range $||\mathbf{G}^T||_h < 5$ (the upper gray plane), remarking that $||\mathbf{G}^T||_h \longrightarrow \infty$ in the neighbourhood of the Riemannian corner, i.e. when $(\eta, \tilde{\eta}) \longrightarrow (1,1)$. The lower plane (gray) represents $||\mathbf{G}^T||_h = 0.5$ which refers to MAT, i.e. $(1,0)$ as well as CROSS, i.e. $(0,1)$. The colors of the related parts in both subfigures correspond to each other.

Finally, we briefly discuss a kind of classification of the navigation problems $\mathcal{P}_{\eta,\tilde{\eta}}$, for any $(\eta, \tilde{\eta}) \in \mathcal{S}$, (Figure 9.3). Taking into account the decompositions of the active wind $\mathbf{G}_{\eta\tilde{\eta}}$, we can give the following classification.

Corollary 9.3.4. *Let $\mathcal{P}_{\eta,\tilde{\eta}}$ be a navigation problem under the action of an active wind $\mathbf{G}_{\eta\tilde{\eta}}$ given in* (9.11)*, on a slippery slope of a mountain (M, h), with a cross-traction coefficient $\eta \in [0,1]$, an along-traction coefficient $\tilde{\eta} \in [0,1]$ and a gravitational wind $\mathbf{G}^T$ on M. The following statements hold:*
i) For any $(\eta, \tilde{\eta}) \in \mathcal{S}$ with $\eta > \tilde{\eta}$, $\mathcal{P}_{\eta,\tilde{\eta}}$ comes from SLIPPERY with a certain form for the cross-traction coefficient, namely $c_1 = \frac{\eta - \tilde{\eta}}{1-\tilde{\eta}} \in (0,1]$;
ii) For any $(\eta, \tilde{\eta}) \in \mathcal{S}$ with $\eta < \tilde{\eta}$, $\mathcal{P}_{\eta,\tilde{\eta}}$ comes from S-CROSS with a certain form for the along-traction coefficient, namely $c_2 = \frac{\tilde{\eta} - \eta}{1-\eta} \in (0,1]$.

Proof. i) Making use of $\mathbf{G}^T = \mathbf{G}_{MAT} + \mathbf{G}^{\perp}_{MAT}$, it yieds $\mathbf{G}_{\eta\tilde{\eta}} = (\tilde{\eta} - \eta)\mathbf{G}^{\perp}_{MAT} + (1 - \tilde{\eta})\mathbf{G}^T$. Since $(\eta, \tilde{\eta}) \in \mathcal{S}$ and $\eta > \tilde{\eta}$, then $\tilde{\eta} \neq 1$. So, we can find out that

$$\mathbf{G}_{\eta\tilde{\eta}} = (1 - \tilde{\eta})[-\frac{\eta - \tilde{\eta}}{1 - \tilde{\eta}}\mathbf{G}^{\perp}_{MAT} + \mathbf{G}^T] = (1 - \tilde{\eta})\mathbf{G}_{c_1},$$

where $\mathbf{G}_{c_1} = c_1\mathbf{G}_{MAT} + (1 - c_1)\mathbf{G}^T$ is the active wind from SLIPPERY with a particular cross-traction coefficient $c_1 = \frac{\eta - \tilde{\eta}}{1-\tilde{\eta}} \in (0,1]$, for any $(\eta, \tilde{\eta}) \in \mathcal{S}$, where $\eta > \tilde{\eta}$. According to Theorem 7.1.1 or [10, Theorem 1.1], the slippery slope metric $\tilde{F}_{c_1}$ corresponding to the active wind $\mathbf{G}_{c_1}$ satisfies the equation

$$\tilde{F}_{c_1}\sqrt{\alpha^2 + 2(1 - c_1)\bar{g}\beta\tilde{F}_{c_1} + (1 - c_1)^2||\mathbf{G}^T||^2_h\tilde{F}^2_{c_1}} = \alpha^2 + (2 - c_1)\bar{g}\beta\tilde{F}_{c_1} + (1 - c_1)||\mathbf{G}^T||^2_h\tilde{F}^2_{c_1}. \tag{9.34}$$

Now, if we substitute $(1-\tilde{\eta})\mathbf{G}^T$ for $\mathbf{G}^T$ everywhere in (9.34) including also $\tilde{F}_{c_1}$, and $c_1 = \frac{\eta-\tilde{\eta}}{1-\tilde{\eta}}$, it turns out that the new $\tilde{F}_{c_1}$ verifies identically (9.29). Thus, we have proved the claim i).

ii) Since $(\eta,\tilde{\eta}) \in \mathcal{S}$ and $\eta < \tilde{\eta}$, then $\eta \neq 1$. To prove ii) we consider S-CROSS with the active wind $\mathbf{G}_{c_2} = -c_2\mathbf{G}_{MAT}+\mathbf{G}^T$, where $c_2 = \frac{\tilde{\eta}-\eta}{1-\eta} \in (0,1]$ is a certain along-traction coefficient. By using Theorem 8.3.1 or [12, Theorem 1.1], the slippery-cross-slope metric $\tilde{F}_{c_2}$ corresponding to $\mathbf{G}_{c_2}$ verifies the equation

$$\tilde{F}_{c_2}\sqrt{\alpha^2 + 2\bar{g}\beta\tilde{F}_{c_2} + ||\mathbf{G}^T||_h^2\tilde{F}_{c_2}^2} = \alpha^2 + (2-c_2)\bar{g}\beta\tilde{F}_{c_2} + (1-c_2)||\mathbf{G}^T||_h^2\tilde{F}_{c_2}^2. \tag{9.35}$$

It is immediate to verify that $\mathbf{G}_{\eta\tilde{\eta}} = (1-\eta)\mathbf{G}_{c_2}$, since $\mathbf{G}_{\eta\tilde{\eta}} = (\eta-\tilde{\eta})\mathbf{G}_{MAT} + (1-\eta)\mathbf{G}^T$. By substituting $(1-\eta)\mathbf{G}^T$ for $\mathbf{G}^T$ everywhere in (9.35) including $\tilde{F}_{c_2}$, and $c_2 = \frac{\tilde{\eta}-\eta}{1-\eta}$, we get that the new $\tilde{F}_{c_2}$ satisfies (9.29). Hence, it follows the required claim. □

We notice that, for any $(\eta,\tilde{\eta}) \in \mathcal{S}$, where $\eta = \tilde{\eta}$, $\mathcal{P}_{\eta,\tilde{\eta}}$ comes from the Zermelo navigation with $\mathbf{G}_{\eta\tilde{\eta}} = (1-\eta)\mathbf{G}^T$, i.e. R-ZNP.

9.3.2 The geodesics of the $(\eta,\tilde{\eta})$-slope metric

Following the strategy presented in recent research [10, 11, 12] (here, Chapters 7 and 8) and basing on some technical computations as well as applying Proposition 6.2.2, we achieve the spray coefficients related to the $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$. By (6.2), it is immediate to supply the equations of time geodesics of $\tilde{F}_{\eta\tilde{\eta}}$. Moreover, the argument that any such time geodesic is unitary with respect to $\tilde{F}_{\eta\tilde{\eta}}$ because before all else, it is a trajectory in Zermelo's navigation developed in Step II, will help us perform the proof of Theorem 9.1.2.

We start by outlining an essential property regarding the $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$. Namely, since $\tilde{F}_{\eta\tilde{\eta}}$ is the root of (9.30), for any $(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}}$, it seems to be a promising general (α,β)-metric. To prove the claim that $\tilde{F}_{\eta\tilde{\eta}}$ is indeed a general (α,β)-metric, let us make the notations: $\tilde{\phi} = \frac{\tilde{F}}{\alpha}$ and $s = \frac{\beta}{\alpha}$. Now, if we divide (9.30) by α^4, we get the equation,

$$\begin{aligned}(1-\eta)^2||\mathbf{G}^T||_h^2[1-(1-\tilde{\eta})^2\,||\mathbf{G}^T||_h^2]\tilde{\phi}^4 + 2(1-\eta)\left[1-(2-\eta-\tilde{\eta})(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right]\bar{g}s\tilde{\phi}^3 \\ +[1-2(1-\eta)(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2 - (2-\eta-\tilde{\eta})^2\,\bar{g}^2s^2]\tilde{\phi}^2 - 2\,(2-\eta-\tilde{\eta})\,\bar{g}s\tilde{\phi} - 1 = 0.\end{aligned} \tag{9.36}$$

This is obviously equivalent to (9.30). Furthermore, since $\tilde{F}_{\eta\tilde{\eta}}$ is the sole positive root of (9.30), it follows that (9.36) also admits a unique positive root, denoted by $\tilde{\phi}_{\eta\tilde{\eta}}$, for any $(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}}$. Pointing out that η and $\tilde{\eta}$ are only parameters, it turns out that $\tilde{\phi}_{\eta\tilde{\eta}}$ depends on the variables $||\mathbf{G}^T||_h^2 = \bar{g}^2b^2$ and $s = \frac{\beta}{\alpha}$, where α and β are given by (9.17), i.e. $\tilde{\phi}_{\eta\tilde{\eta}} = \tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2, s)$ and also, $\tilde{\phi}_{\eta\tilde{\eta}}$ is a positive C^∞-function because $\tilde{F}_{\eta\tilde{\eta}}(x,y) = \alpha\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2, s)$. Thus, the requested claim is proved.

There are still some emerging properties regarding the function $\tilde{\phi}_{\eta\tilde{\eta}}$ as well as its derivatives. An essential role in our study is played by the following identity

$$\begin{aligned}(1-\eta)^2||\mathbf{G}^T||_h^2[1-(1-\tilde{\eta})^2\,||\mathbf{G}^T||_h^2]\tilde{\phi}_{\eta\tilde{\eta}}^4 + 2(1-\eta)\left[1-(2-\eta-\tilde{\eta})(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right]\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}}^3 \\ +\{\left[1-2(1-\eta)(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right]\alpha^2 - (2-\eta-\tilde{\eta})^2\,\bar{g}^2s^2\}\tilde{\phi}_{\eta\tilde{\eta}}^2 - 2\,(2-\eta-\tilde{\eta})\,\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}} - 1 = 0,\end{aligned} \tag{9.37}$$

which follows from the fact that $\tilde{\phi}_{\eta\tilde{\eta}}$ checks identically (9.36), for any $(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}}$. Having the inequality $||\mathbf{G}^T||_h < \tilde{b}_0$, with $\tilde{b}_0$ defined in (9.33), which secures the strong convexity of the indicatrix $I_{\tilde{F}_{\eta\tilde{\eta}}}$ according to Theorem 9.1.1, we can apply the direct implication of Proposition 6.2.1. Hence, for any $(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}}$ and s satisfying $|s| \leq \frac{||\mathbf{G}^T||_h}{\bar{g}} < \frac{\tilde{b}_0}{\bar{g}}$, we have guaranteed the validity of the following inequalities

$$\tilde{\phi}_{\eta\tilde{\eta}} - s\tilde{\phi}_{\eta\tilde{\eta}2} > 0, \qquad \bar{g}^2(\tilde{\phi}_{\eta\tilde{\eta}} - s\tilde{\phi}_{\eta\tilde{\eta}2}) + (||\mathbf{G}^T||_h^2 - \bar{g}^2 s^2)\tilde{\phi}_{\eta\tilde{\eta}22} > 0,$$

when $n \geq 3$, or only the right-hand side inequality, when $n = 2$.

Lemma 9.3.5. *Let M be an n-dimensional manifold, $n > 1$, with the $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}} = \alpha\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2, s)$. For any $(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}}$, the function $\tilde{\phi}_{\eta\tilde{\eta}}$ and its derivative with respect to s, i.e. $\tilde{\phi}_{\eta\tilde{\eta}2}$ hold the following relations:*

$$C\tilde{\phi}_{\eta\tilde{\eta}2} = \bar{g}A\tilde{\phi}_{\eta\tilde{\eta}}, \quad C(\tilde{\phi}_{\eta\tilde{\eta}} - s\tilde{\phi}_{\eta\tilde{\eta}2}) = B, \quad C\tilde{\phi}_{\eta\tilde{\eta}} = B + \bar{g}sA\tilde{\phi}_{\eta\tilde{\eta}}, \quad (2-\eta-\tilde{\eta})\,B - 2A = (\tilde{\eta}-\eta)\tilde{\phi}_{\eta\tilde{\eta}}^2, \tag{9.38}$$

where

$$\begin{aligned}
A &= -(1-\eta)\left[1-(2-\eta-\tilde{\eta})(1-\tilde{\eta})||\mathbf{G}^T||_h^2\right]\tilde{\phi}_{\eta\tilde{\eta}}^2 + (2-\eta-\tilde{\eta})^2\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}} + 2-\eta-\tilde{\eta},\\
B &= -\left[1-2(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2\right]\tilde{\phi}_{\eta\tilde{\eta}}^2 + 2(2-\eta-\tilde{\eta})\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}} + 2,\\
C &= 2(1-\eta)^2||\mathbf{G}^T||_h^2\left[1-(1-\tilde{\eta})^2||\mathbf{G}^T||_h^2\right]\tilde{\phi}_{\eta\tilde{\eta}}^3 + 3(1-\eta)\left[1-(2-\eta-\tilde{\eta})(1-\tilde{\eta})||\mathbf{G}^T||_h^2\right]\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}}^2\\
&\quad + \{\left[1-2(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2\right] - (2-\eta-\tilde{\eta})^2\bar{g}^2s^2\}\tilde{\phi}_{\eta\tilde{\eta}} - (2-\eta-\tilde{\eta})\bar{g}s
\end{aligned} \tag{9.39}$$

and A, B, C are evaluated at $(||\mathbf{G}^T||_h^2, s)$.

Proof. By differentiating the identity (9.37) with respect to s, it follows the first relation in (9.38). The proof of the second identity in (9.38) is based on the first one and on some simple computations. Finally, by using the notations (9.39) and (9.37), it turns out the last two relations in (9.38). □

Lemma 9.3.6. *Let M be an n-dimensional manifold, $n > 1$, with the $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}} = \alpha\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2, s)$. For any $(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}}$ and s such that $|s| \leq \frac{||\mathbf{G}^T||_h}{\bar{g}} < \frac{\tilde{b}_0}{\bar{g}}$, the following statements hold:*
i) $C(||\mathbf{G}^T||_h^2, s) \neq 0$, $\tilde{\phi}_{\eta\tilde{\eta}2} = \frac{\bar{g}A}{C}\tilde{\phi}_{\eta\tilde{\eta}}$, and $\tilde{\phi}_{\eta\tilde{\eta}} - s\tilde{\phi}_{\eta\tilde{\eta}2} = \frac{B}{C}$.
ii) $B(||\mathbf{G}^T||_h^2, s) \neq 0$.

Proof. i) Clearly, if $\eta = \tilde{\eta} = 1$, then $C = \tilde{\phi}_{\eta\tilde{\eta}} > 0$. Now we prove that $C(||\mathbf{G}^T||_h^2, s) \neq 0$, for any $(\eta,\tilde{\eta}) \in \mathcal{S}$. We assume by contradiction that there exists $s_0 \in [-b, b]$, $b = \frac{||\mathbf{G}^T||_h}{\bar{g}} < \frac{\tilde{b}_0}{\bar{g}}$, with $\tilde{b}_0$ defined by (9.33), such that $C(||\mathbf{G}^T||_h^2, s_0) = 0$. With this assumption, due to (9.38), we get

$$A(||\mathbf{G}^T||_h^2, s_0) = B(||\mathbf{G}^T||_h^2, s_0) = (\eta-\tilde{\eta})\tilde{\phi}_{\eta\tilde{\eta}}^2(||\mathbf{G}^T||_h^2, s_0) = 0. \tag{9.40}$$

Since $\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2, s_0) > 0$ for any $(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}}$, the last equality in (9.40) turns out that the only possibility is that $\eta = \tilde{\eta} \neq 1$, i.e. $(\eta,\tilde{\eta}) \in \mathcal{L}_0$. By using this fact and (9.40), the identity (5.4) is reduced to

$$(1-\eta)^2||\mathbf{G}^T||_h^2[1-(1-\eta)^2||\mathbf{G}^T||_h^2]\tilde{\phi}_{\eta\tilde{\eta}}^4 + [2(1-\eta)\bar{g}s_0\tilde{\phi}_{\eta\tilde{\eta}} + 1]^2 = 0, \tag{9.41}$$

where $\tilde{\phi}_{\eta\tilde{\eta}}$ is evaluated at $(||\mathbf{G}^T||_h^2, s_0)$ and it is with $\eta=\tilde{\eta}\neq 1$. As $||\mathbf{G}^T||_h < \frac{1}{1-\eta}$ for $(\eta,\tilde{\eta})\in\mathcal{L}_0$, (9.41) contradicts the fact that $(1-\eta)^2||\mathbf{G}^T||_h^2[1-(1-\eta)^2||\mathbf{G}^T||_h^2]\tilde{\phi}^4_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2, s_0)>0$. Thus, we have shown that $C\neq 0$ everywhere. Moreover, making use of (9.38), the claims i) are fulfilled.

ii) If $\eta=\tilde{\eta}=1$, then $B=\tilde{\phi}^2_{\eta\tilde{\eta}}=1$. However, it remains to prove that $B(||\mathbf{G}^T||_h^2, s)\neq 0$, for any $(\eta,\tilde{\eta})\in\mathcal{S}$. We assume, towards a contradiction, that there is $\tilde{s}\in[-b,b]$, $b=\frac{||\mathbf{G}^T||_h}{\bar{g}}<\frac{\tilde{b}_0}{\bar{g}}$, with $\tilde{b}_0$ defined by (9.33), such that $B(||\mathbf{G}^T||_h^2,\tilde{s})=0$. So, we are searching now for such an $\tilde{s}$.

On one hand, since $\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2,\tilde{s})>0$, $C(||\mathbf{G}^T||_h^2,\tilde{s})\neq 0$ and $B(||\mathbf{G}^T||_h^2,\tilde{s})=0$, the third formula in (9.38) with $s=\tilde{s}$, implies that $\tilde{s}\neq 0$. On the other hand, due to our assumption, by the second formula in (9.39), it follows that $\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2,\tilde{s})$ satisfies the polynomial equation

$$[1-2(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2]\tilde{\phi}^2_{\eta\tilde{\eta}}-2(2-\eta-\tilde{\eta})\bar{g}\tilde{s}\tilde{\phi}_{\eta\tilde{\eta}}-2=0. \tag{9.42}$$

Moreover, for $s=\tilde{s}$ and for any $(\eta,\tilde{\eta})\in\mathcal{S}$, (9.37) is reduced to

$$\begin{gathered}2(1-\eta)^2\zeta||\mathbf{G}^T||_h^2\tilde{\phi}^2_{\eta\tilde{\eta}}+\left[2-3\eta+\tilde{\eta}-2\left(2-\eta-\tilde{\eta}\right)\left(1-\eta\right)\left(1-\tilde{\eta}\right)||\mathbf{G}^T||_h^2\right]\bar{g}\tilde{s}\tilde{\phi}_{\eta\tilde{\eta}}\\ +1-2(1-\eta)\left(1-\tilde{\eta}\right)||\mathbf{G}^T||_h^2=0,\end{gathered} \tag{9.43}$$

where ζ denotes the term $1-(1-\tilde{\eta})^2\,||\mathbf{G}^T||_h^2$.

Since $||\mathbf{G}^T||_h<\tilde{b}_0$, with $\tilde{b}_0$ defined by (9.33), it turns out that $\zeta=1-(1-\tilde{\eta})^2\,||\mathbf{G}^T||_h^2\neq 0$ for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$. Nevertheless, there may exist some $(\eta,\tilde{\eta})\in\mathcal{S}\setminus(\mathcal{L}_1\cup\mathcal{L}_2)$ such that $1-2(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2=0$. Thus, we have to analyze two cases.

Case a. If $1-2(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2\neq 0$, for any $(\eta,\tilde{\eta})\in\mathcal{S}$, then due to (9.42) and (9.43), we get

$$[1+4(1-\eta)\left(\tilde{\eta}-\eta\right)||\mathbf{G}^T||_h^2]\tilde{\phi}_{\eta\tilde{\eta}}+4\left(\tilde{\eta}-\eta\right)\bar{g}\tilde{s}=0. \tag{9.44}$$

The last equation provides a contradiction when $(\eta,\tilde{\eta})\in\mathcal{L}_0$. Thus, $\eta\neq\tilde{\eta}$ and moreover, since $\tilde{s}\neq 0$ and $\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2,\tilde{s})>0$ it turns out that $1+4(1-\eta)\left(\tilde{\eta}-\eta\right)||\mathbf{G}^T||_h^2\neq 0$ and

$$\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2,\tilde{s})=-\frac{4\left(\tilde{\eta}-\eta\right)\bar{g}\tilde{s}}{1+4(1-\eta)\left(\tilde{\eta}-\eta\right)||\mathbf{G}^T||_h^2}. \tag{9.45}$$

Once we have (9.45) for any $(\eta,\tilde{\eta})\in\mathcal{S}\setminus\mathcal{L}_0$, we can go with it in (9.42) and the result is

$$4\left(\tilde{\eta}-\eta\right)\bar{g}^2\tilde{s}^2[2-3\eta+\tilde{\eta}+4(\tilde{\eta}-\eta)(1-\eta)^2||\mathbf{G}^T||_h^2]=[1+4(1-\eta)\left(\tilde{\eta}-\eta\right)||\mathbf{G}^T||_h^2]^2.$$

Since the right-hand side of this is positive, it follows that

$$\left(\tilde{\eta}-\eta\right)[2-3\eta+\tilde{\eta}+4(\tilde{\eta}-\eta)(1-\eta)^2||\mathbf{G}^T||_h^2]>0$$

and thus, we obtain $\tilde{s}^2=\frac{[1+4(1-\eta)(\tilde{\eta}-\eta)||\mathbf{G}^T||_h^2]^2}{4(\tilde{\eta}-\eta)\bar{g}^2[2-3\eta+\tilde{\eta}+4(\tilde{\eta}-\eta)(1-\eta)^2||\mathbf{G}^T||_h^2]}$, which contradicts $\tilde{s}^2\in(0,b^2]$, for any $(\eta,\tilde{\eta})\in\mathcal{S}\setminus\mathcal{L}_0$, due to the condition $||\mathbf{G}^T||_h<\tilde{b}_0$, where $\tilde{b}_0$ is defined by (9.33). Indeed, since we must have $\tilde{s}^2\leq b^2$, this implies $||\mathbf{G}^T||_h\geq\frac{1}{2|\eta-\tilde{\eta}|}$ for any $(\eta,\tilde{\eta})\in\mathcal{S}\setminus\mathcal{L}_0$, noticing that $\frac{1}{2|\eta-\tilde{\eta}|}>\frac{1}{1-\tilde{\eta}}$ for any $(\eta,\tilde{\eta})\in\mathcal{D}_1\cup\mathcal{D}_2\setminus\mathcal{L}_0$.

Case b. If $1-2(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2=0$, for some $(\eta,\tilde{\eta})\in\mathcal{S}\smallsetminus(\mathcal{L}_1\cup\mathcal{L}_2)$, then, by (9.42), it follows that

$$\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2,\tilde{s})=-\frac{1}{(2-\eta-\tilde{\eta})\bar{g}\tilde{s}}. \tag{9.46}$$

The last equation together with (9.43) lead to $\tilde{s}$ which satisfies the relation

$$\tilde{s}^2\,(\tilde{\eta}-\eta)=\frac{1-2\eta+\tilde{\eta}}{4(1-\tilde{\eta})(2-\eta-\tilde{\eta})\bar{g}^2}. \tag{9.47}$$

Obviously, when $\eta=\tilde{\eta}$, (9.47) provides a contradiction. Therefore, it remains to study (9.47) when $(\eta,\tilde{\eta})\in\mathcal{S}\smallsetminus(\bigcup_{i=0}^{2}\mathcal{L}_i)$. Since $\tilde{s}\neq 0$ and $\eta\neq\tilde{\eta}$, it follows that $1-2\eta+\tilde{\eta}\neq 0$. Now, making use of $\tilde{s}^2\leq b^2=\frac{||\mathbf{G}^T||_h^2}{\bar{g}^2}$, it turns out that there exist $(\eta,\tilde{\eta})\in\tilde{\mathcal{D}}\subset\mathcal{S}\smallsetminus(\bigcup_{i=0}^{2}\mathcal{L}_i)$ such that $1-2(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2=0$, where $\tilde{\mathcal{D}}=\tilde{\mathcal{D}}_3\cup\tilde{\mathcal{D}}_4$ and

$$\tilde{\mathcal{D}}_3=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid \tfrac{1}{2}<\eta<1,\ \tilde{\eta}<2\eta-1\right\}\subset\mathcal{D}_3,$$

$$\tilde{\mathcal{D}}_4=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid \eta\leq 2\tilde{\eta}-1,\ \tfrac{1}{2}\leq\tilde{\eta}<1\right\}\subset\mathcal{D}_4.$$

However, the fact that $||\mathbf{G}^T||_h<\frac{1}{2|\eta-\tilde{\eta}|}$ on $\tilde{\mathcal{D}}$ is contradicted.

Summing up the above findings, we have proved that $B(||\mathbf{G}^T||_h^2,s)\neq 0$, for any $s\in[-b,b]$, $b=\frac{||\mathbf{G}^T||_h}{\bar{g}}<\frac{\tilde{b}_0}{\bar{g}}$, with $\tilde{b}_0$ defined by (9.33). □

We remark that basing on Proposition 6.2.1 it is known that $\tilde{\phi}_{\eta\tilde{\eta}}-s\tilde{\phi}_{\eta\tilde{\eta}2}>0$, when $n\geq 3$, for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$ and s such that $|s|\leq\frac{||\mathbf{G}^T||_h}{\bar{g}}<\frac{\tilde{b}_0}{\bar{g}}$. Now, according to Lemma 9.3.5, since $\tilde{\phi}_{\eta\tilde{\eta}}-s\tilde{\phi}_{\eta\tilde{\eta}2}=\frac{B}{C}$ we have proved that $\tilde{\phi}_{\eta\tilde{\eta}}-s\tilde{\phi}_{\eta\tilde{\eta}2}\neq 0$ also when $n=2$, for any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$ and $|s|\leq\frac{||\mathbf{G}^T||_h}{\bar{g}}<\frac{\tilde{b}_0}{\bar{g}}$.

Lemma 9.3.7. *Let M be an n-dimensional manifold, $n>1$, with the $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}=\alpha\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2,s)$. For any $(\eta,\tilde{\eta})\in\tilde{\mathcal{S}}$, the first order derivative of the function $\tilde{\phi}_{\eta\tilde{\eta}}$ with respect to $b^2=\frac{||\mathbf{G}^T||_h^2}{\bar{g}^2}$, i.e. $\tilde{\phi}_{\eta\tilde{\eta}1}$ and the second order derivatives $\tilde{\phi}_{\eta\tilde{\eta}12}$ and $\tilde{\phi}_{\eta\tilde{\eta}22}$ hold the following relations:*

$$\begin{aligned}
\tilde{\phi}_{\eta\tilde{\eta}1}&=\tfrac{(1-\eta)\bar{g}^2}{2C}[(1-\tilde{\eta})\,B-(\tilde{\eta}-\eta)\tilde{\phi}^2_{\eta\tilde{\eta}}]\tilde{\phi}^2_{\eta\tilde{\eta}},\\
\tilde{\phi}_{\eta\tilde{\eta}12}&=\tfrac{(1-\eta)\bar{g}^3}{2C^3}\left\{A(B+C\tilde{\phi}_{\eta\tilde{\eta}})[(1-\tilde{\eta})\,B-(\tilde{\eta}-\eta)\tilde{\phi}^2_{\eta\tilde{\eta}}]+(\tilde{\eta}-\eta)^2[2+(1-\eta)\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}}]\tilde{\phi}^4_{\eta\tilde{\eta}}\right\}\tilde{\phi}_{\eta\tilde{\eta}},\\
\tilde{\phi}_{\eta\tilde{\eta}22}&=\tfrac{\bar{g}^2}{C^3}[A^2B+(\tilde{\eta}-\eta)^2\tilde{\phi}^4_{\eta\tilde{\eta}}].
\end{aligned} \tag{9.48}$$

Proof. By differentiating the identity (9.37) with respect to $||\mathbf{G}^T||_h^2$, we get

$$\frac{\partial\tilde{\phi}_{\eta\tilde{\eta}}}{\partial||\mathbf{G}^T||_h^2}=\frac{1-\eta}{2C}[(1-\tilde{\eta})B-(\tilde{\eta}-\eta)\tilde{\phi}^2_{\eta\tilde{\eta}}]\tilde{\phi}^2_{\eta\tilde{\eta}}.$$

Now, if we substitute this into $\tilde{\phi}_{\eta\tilde{\eta}1} = \bar{g}^2 \frac{\partial\tilde{\phi}_{\eta\tilde{\eta}}}{\partial||\mathbf{G}^T||_h^2}$, the first relation in (9.48) follows. Differentiating the functions (9.39) with respect to s, together with (9.38) yield the following identities

$$\begin{aligned} A_2 &= \frac{\bar{g}}{C}[2A^2 + (2-\eta-\tilde{\eta})(\tilde{\eta}-\eta)\tilde{\phi}^2_{\eta\tilde{\eta}}], \qquad B_2 = \frac{2\bar{g}}{C}[AB + (\tilde{\eta}-\eta)\tilde{\phi}^2_{\eta\tilde{\eta}}], \\ C_2 &= -\frac{\bar{g}}{C\tilde{\phi}_{\eta\tilde{\eta}}}\{AB - (\tilde{\eta}-\eta)[2 + (2-\eta-\tilde{\eta})\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}}]\tilde{\phi}^2_{\eta\tilde{\eta}}\} + 3\bar{g}A, \end{aligned}$$

where $A_2 = \frac{\partial A}{\partial s}$, $B_2 = \frac{\partial B}{\partial s}$ and $C_2 = \frac{\partial B}{\partial s}$. All these, along with

$$\tilde{\phi}_{\eta\tilde{\eta}12} = \tfrac{(1-\eta)\bar{g}^2}{2C^2}\{(1-\tilde{\eta})B_2C + 2\bar{g}A[(1-\tilde{\eta})\,B - 2(\tilde{\eta}-\eta)\tilde{\phi}^2_{\eta\tilde{\eta}}] - [(1-\tilde{\eta})\,B - (\tilde{\eta}-\eta)\tilde{\phi}^2_{\eta\tilde{\eta}}]C_2\}\tilde{\phi}^2_{\eta\tilde{\eta}},$$

$$\tilde{\phi}_{\eta\tilde{\eta}22} = \tfrac{\bar{g}}{C^2}(A_2C + \bar{g}A^2 - AC_2)\tilde{\phi}_{\eta\tilde{\eta}},$$

give the last two formulas in (9.48). □

Now we are in position to provide the spray coefficients corresponding to the general (α,β)-metric $\tilde{F}_{\eta\tilde{\eta}}$.

Lemma 9.3.8. *Let M be an n-dimensional manifold, $n > 1$, with the $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$, having a cross-traction coefficient $\eta \in [0,1]$ and an along-traction coefficient $\tilde{\eta} \in [0,1]$. Then, the relationship between the spray coefficients $\tilde{\mathcal{G}}^i_{\eta\tilde{\eta}}$ of $\tilde{F}_{\eta\tilde{\eta}}$ and the spray coefficients $\mathcal{G}^i_\alpha = \frac{1}{4}h^{im}\left(2\frac{\partial h_{jm}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^m}\right)y^jy^k$ of α, is given by*

$$\tilde{\mathcal{G}}^i_{\eta\tilde{\eta}}(x,y) = \mathcal{G}^i_\alpha(x,y) + \left[\Theta(r_{00} + 2\alpha^2Rr) + \alpha\Omega r_0\right]\frac{y^i}{\alpha} - \left[\Psi(r_{00} + 2\alpha^2Rr) + \alpha\Pi r_0\right]\frac{w^i}{\bar{g}} - \alpha^2Rr^i, \tag{9.49}$$

$i = 1,...,n$, where

$$r_{00} = -\tfrac{1}{\bar{g}}w_{i|j}y^iy^j, \quad r_0 = \tfrac{1}{\bar{g}^2}w_{i|j}w^jy^i, \quad r = -\tfrac{1}{\bar{g}^3}w_{i|j}w^iw^j, \quad r^i = \tfrac{1}{\bar{g}^2}{w^i}_{|j}w^j,$$

$$\begin{aligned} R &= \tfrac{(1-\eta)\bar{g}^2}{2\alpha^4B}[(1-\tilde{\eta})\,\alpha^2B - (\tilde{\eta}-\eta)\,\tilde{F}^2_{\tilde{\eta}}]\tilde{F}^2_{\eta\tilde{\eta}}, \\ \Theta &= \tfrac{\bar{g}\alpha}{2E\tilde{F}_{\eta\tilde{\eta}}}[\alpha^6AB^2 - (\tilde{\eta}-\eta)^2\,\bar{g}\beta\tilde{F}^5_{\eta\tilde{\eta}}], \quad \Psi = \tfrac{\bar{g}^2\alpha^2}{2E}[\alpha^4A^2B + (\tilde{\eta}-\eta)^2\,\tilde{F}^4_{\eta\tilde{\eta}}], \\ \Omega &= \tfrac{(1-\eta)\bar{g}^2}{\alpha^2BE}\left\{[(1-\tilde{\eta})\,\alpha^2B - (\tilde{\eta}-\eta)\,\tilde{F}^2_{\eta\tilde{\eta}}][\alpha^6B^3 + (\tilde{\eta}-\eta)^2\,||\mathbf{G}^T||^2_h\tilde{F}^6_{\eta\tilde{\eta}}]\right. \\ &\quad \left. - (\tilde{\eta}-\eta)^2\,\alpha^2\tilde{F}^5_{\eta\tilde{\eta}}(\bar{g}\beta B + ||\mathbf{G}^T||^2_hA\tilde{F}_{\eta\tilde{\eta}})\right\}, \\ \Pi &= \tfrac{(1-\eta)\bar{g}^3}{2\alpha^3BE}\left\{[(1-\tilde{\eta})\,\alpha^2B - (\tilde{\eta}-\eta)\,\tilde{F}^2_{\eta\tilde{\eta}}][2\alpha^6AB^2 - (\tilde{\eta}-\eta)^2\,\bar{g}\beta\tilde{F}^5_{\eta\tilde{\eta}}]\right. \\ &\quad \left. + (\tilde{\eta}-\eta)^2\,\alpha^2B\tilde{F}^4_{\eta\tilde{\eta}}[2\alpha^2 + (1-\eta)\,\bar{g}\beta\tilde{F}_{\eta\tilde{\eta}}]\right\}\tilde{F}_{\eta\tilde{\eta}}, \end{aligned} \tag{9.50}$$

with

$$A = -\frac{1}{\alpha^2}\{(1-\eta)\left[1-(2-\eta-\tilde{\eta})(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right]\tilde{F}^2_{\eta\tilde{\eta}} - (2-\eta-\tilde{\eta})^2\bar{g}\beta\tilde{F}_{\eta\tilde{\eta}} - (2-\eta-\tilde{\eta})\alpha^2\},$$

$$B = -\frac{1}{\alpha^2}\{[1-2(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2]\tilde{F}^2_{\eta\tilde{\eta}} - 2(2-\eta-\tilde{\eta})\bar{g}\beta\tilde{F}_{\eta\tilde{\eta}} - 2\alpha^2\},$$

$$C = \frac{1}{\alpha\tilde{F}_{\eta\tilde{\eta}}}\left(\alpha^2 B + \bar{g}\beta A\tilde{F}_{\eta\tilde{\eta}}\right),$$

$$E = \alpha^6 BC^2 + (||\mathbf{G}^T||_h^2\alpha^2 - \bar{g}^2\beta^2)[\alpha^4 A^2 B + (\eta-\tilde{\eta})^2\tilde{F}^4_{\eta\tilde{\eta}}]. \tag{9.51}$$

Proof. Having the derivatives $\tilde{\phi}_{\eta\tilde{\eta}1}$, $\tilde{\phi}_{\eta\tilde{\eta}2}$, $\tilde{\phi}_{\eta\tilde{\eta}12}$ and $\tilde{\phi}_{\eta\tilde{\eta}22}$ given by Lemma 9.3.6 i) and (9.48), a simple computation shows that

$$s\tilde{\phi}_{\eta\tilde{\eta}} + (b^2-s^2)\tilde{\phi}_{\eta\tilde{\eta}2} = \frac{1}{\bar{g}C}(\bar{g}sB + ||\mathbf{G}^T||_h^2 A\tilde{\phi}_{\eta\tilde{\eta}}),$$

$$(\tilde{\phi}_{\eta\tilde{\eta}} - s\tilde{\phi}_{\eta\tilde{\eta}2})\tilde{\phi}_{\eta\tilde{\eta}2} - s\tilde{\phi}_{\eta\tilde{\eta}}\tilde{\phi}_{\eta\tilde{\eta}22} = \frac{\bar{g}}{C^3}[AB^2 - (\tilde{\eta}-\eta)^2\bar{g}s\tilde{\phi}^5_{\eta\tilde{\eta}}],$$

$$\tilde{\phi}_{\eta\tilde{\eta}} - s\tilde{\phi}_{\eta\tilde{\eta}2} + (b^2-s^2)\tilde{\phi}_{\eta\tilde{\eta}22} = \frac{1}{C^3}\{BC^2 + (||\mathbf{G}^T||_h^2 - \bar{g}^2s^2)[A^2B + (\tilde{\eta}-\eta)^2\tilde{\phi}^4_{\eta\tilde{\eta}}]\},$$

$$(\tilde{\phi}_{\eta\tilde{\eta}} - s\tilde{\phi}_{\eta\tilde{\eta}2})\tilde{\phi}_{\eta\tilde{\eta}12} - s\tilde{\phi}_{\eta\tilde{\eta}1}\tilde{\phi}_{\eta\tilde{\eta}22} = \frac{(1-\eta)\bar{g}^3}{2C^4}\{[(1-\tilde{\eta})B - (\tilde{\eta}-\eta)\tilde{\phi}^2_{\eta\tilde{\eta}}][2AB^2 - (\tilde{\eta}-\eta)^2\bar{g}s\tilde{\phi}^5_{\eta\tilde{\eta}}]$$
$$+ (\tilde{\eta}-\eta)^2[2+(1-\eta)\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}}]B\tilde{\phi}^4_{\eta\tilde{\eta}}\}\tilde{\phi}_{\eta\tilde{\eta}}. \tag{9.52}$$

Denoting by w^i the components of $\mathbf{G}^T = -\bar{g}h^{ji}\frac{\partial p}{\partial x^j}\frac{\partial}{\partial x^i}$ and using the notation $w_i = h_{ij}w^j$, it follows that $w_i = -\bar{g}\frac{\partial p}{\partial x^i}$ and $\frac{\partial w_i}{\partial x^j} = \frac{\partial w_j}{\partial x^i}$. Moreover, according to [10, Lemma 4.3], we have $s_{ij} = s_i = s^i = s^i_0 = s_0 = 0$ as well as the the relations 7.39. Collecting the findings (9.52) and (7.39), one can apply Proposition 6.2.2, and thus, our claim yields at once. □

We notice that a simplified form of the spray coefficients $\tilde{\mathcal{G}}^i_{\eta\tilde{\eta}}(x,y)$ occurs when $||\mathbf{G}^T||_h$ is constant. Indeed, making use of [10, Lemma 4.3] again, since $\frac{\partial||\mathbf{G}^T||_h}{\partial x^i} = \frac{2}{\bar{g}^2}w_{i|j}w^j = 2r_i$, we clearly have that $r_i = 0$ if and only if $||\mathbf{G}^T||_h$ is constant and furthermore, the statement $r_i = 0$ implies $r^i = r = r_0 = 0$. All these particularities reduce the formula (9.49) to (7.43) with (9.49).

It remains only to catch the ODE system which provides the shortest time trajectories $\gamma(t) = (\gamma^i(t))$, $i = 1, ..., n$ on the slippery slope, under the influence of the active wind $\mathbf{G}_{\eta\tilde{\eta}}$. Namely, if we substitute the spray coefficients $\tilde{\mathcal{G}}^i_{\eta\tilde{\eta}}(\gamma(t),\dot{\gamma}(t))$ from (9.49) into (6.2), with $\tilde{F}_{\eta\tilde{\eta}}(\gamma(t),\dot{\gamma}(t)) = 1$, it turns out the system (9.2). This ends the proof of Theorem 9.1.2.

Finally, we provide two examples which support the applicability of the above obtained results by highlighting the two-dimensional case.

Example 1. We start with an inclined plane (a ramp) because this example allows us to show clearly the behaviour of the indicatrix of the $(\eta,\tilde{\eta})$-slope metrics $\tilde{F}_{\eta\tilde{\eta}}$, for any pair $(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}}$. We consider the planar slope given by $z = x/2$ (i.e. $f(x^1,x^2) = x/2$, where $x = x^1$, $y = x^2$) having the slope angle 26.6° and taking the regular point $O = (0,0)$ as the center of the indicatrix. In this setting, it turns out that $h = \sqrt{h_{ij}y^iy^j}$ has $h_{11} = 5/4$, $h_{22} = 1$,

$h_{12} = h_{21} = 0$ as well as $q = 1/4$, $\mathbf{G}^T = -\frac{2\bar{g}}{5}\frac{\partial}{\partial x^1}$ and $||\mathbf{G}^T||_h = \frac{\bar{g}}{\sqrt{5}}$. Moreover, it follows that $y^1 = -2X/\sqrt{5}$ and $y^2 = -Y$ and the equation of motions are given by [13]

$$\begin{cases} -\frac{\sqrt{5}}{2}y^1 & = & [1 + (\eta - \tilde{\eta})\frac{\bar{g}}{\sqrt{5}}\cos\theta]\cos\theta + (1-\eta)\frac{\bar{g}}{\sqrt{5}} \\ -y^2 & = & [1 + (\eta - \tilde{\eta})\frac{\bar{g}}{\sqrt{5}}\cos\theta]\sin\theta \end{cases}, \qquad (9.53)$$

for any direction $\theta \in [0, 2\pi)$ of the velocity u. By applying the general theory presented in the previous sections, the strong convexity condition $||\mathbf{G}^T||_h < \tilde{b}_0$, with $\tilde{b}_0$ defined in (9.33), which corresponds to the inclined plane, is equivalent to $\bar{g} < \delta_1(\eta, \tilde{\eta})$, where

$$\delta_1(\eta, \tilde{\eta}) = \begin{cases} \frac{\sqrt{5}}{1-\tilde{\eta}}, & \text{if} \quad (\eta, \tilde{\eta}) \in \mathcal{D}_1 \cup \mathcal{D}_2 \\ \frac{\sqrt{5}}{2|\eta - \tilde{\eta}|}, & \text{if} \quad (\eta, \tilde{\eta}) \in \mathcal{D}_3 \cup \mathcal{D}_4 \end{cases}. \qquad (9.54)$$

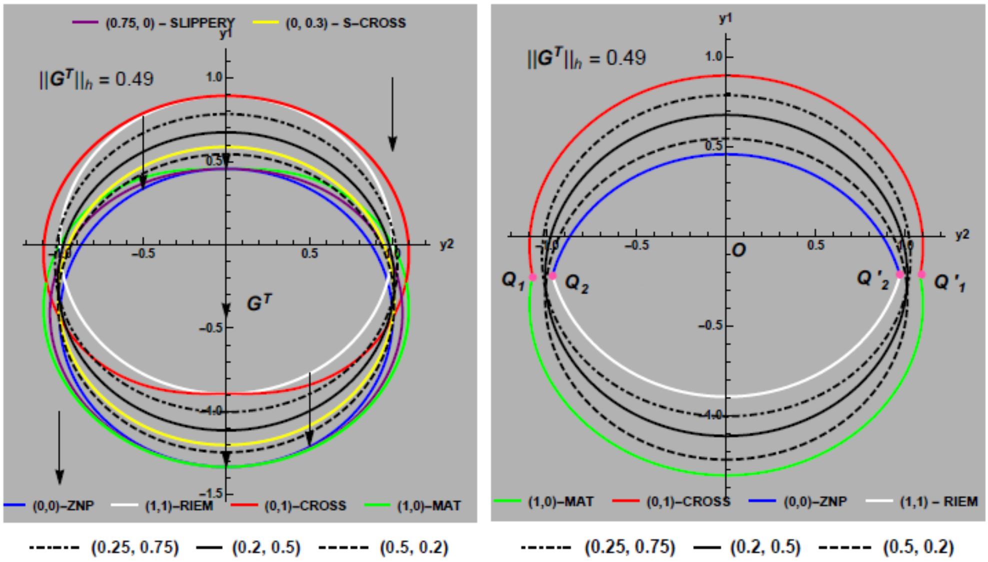


Figure 9.4: Left: The comparison of all specific types of the Finslerian indicatrices (the colour-coded limaçons) centered at the origin of the coordinate system y^1Oy^2 on the planar $(\eta, \tilde{\eta})$-slope given by $z = x/2$, under the action of $\mathbf{G}^T$ (indicated by black arrows) of constant force $||\mathbf{G}^T||_h = 0.49$; $t = 1$. The steepest downhill direction is indicated by the negative axis y^1. Right: All $(\eta, \tilde{\eta})$-indicatrices (black) are located between the boundaries consisting of MAT (the lower part, green) and CROSS (the upper part, red), i.e. the maximum range, as well as ZNP (the upper part, blue) and RIEM (the lower part, white), i.e. the minimum range. The MAT and CROSS indicatrices intersect each other in the points Q_1 and Q_1', which correspond to the directions of the self-velocity u: $\theta_{MAT} \in \{77.2°, 282.8°\}$ and $\theta_{CROSS} \in \{102.8°, 257.2°\}$ or, equivalently, the directions of the resultant velocity $v_{\eta\tilde{\eta}}$: $\tilde{\theta} \in \{77.2°, 282.8°\}$, respectively, and $||v_{\eta\tilde{\eta}}||_h \approx 1.108$, where $||u||_h = 1$. The ZNP and RIEM indicatrices intersect each other in the points Q_2 and Q_2', which correspond to the directions of the self-velocity u: $\theta_{RIEM} \in \{75.8°, 284.2°\}$ and $\theta_{ZNP} \in \{104.2°, 255.8°\}$ or, equivalently, the directions of the resultant velocity $v_{\eta\tilde{\eta}}$: $\tilde{\theta} \in \{75.8°, 284.2°\}$, respectively, and $||v_{\eta\tilde{\eta}}||_h = 1$.

We compare all specific types of the Finslerian indicatrices considered in our study in Figure 9.4 (left), i.e. ZNP, MAT, CROSS, RIEM, SLIPPERY, S-CROSS and three new cases coming from the interior of the problem square diagram $\tilde{\mathcal{S}}$, i.e. $(0.25, 0.75), (0.2, 0.5)$ and $(0.5, 0.2)$. The force of the gravitational wind blowing on the planar slope under consideration equals 0.49, which is due to the conditions for strong convexity in the most stringent cases, i.e. MAT and CROSS, where $||\mathbf{G}^T||_h < 0.5$. Therefore, $\bar{g} < \sqrt{5}/2 \approx 1.118$, since $||\mathbf{G}^T||_h = \bar{g}/\sqrt{5}$ for the ramp $z = x/2$. Interestingly, the maximum range of an arbitrary $(\eta, \tilde{\eta})$-indicatrix in any direction is created by MAT and CROSS as well as the minimum range by ZNP and RIEM. Namely, all $(\eta, \tilde{\eta})$-indicatrices are located in between those boundaries; for the sake of clarity, see Figure 9.4 (right) in this regard.

Example 2. We consider a triple Gaussian bell-shaped hill $\mathfrak{G}_3$ given by the function

$$z = f(x_1, x_2) = \frac{1}{4}\sum_{k=1}^{3}(k+1)e^{-\rho_k} = \frac{1}{2}e^{-\rho_1} + \frac{3}{4}e^{-\rho_2} + e^{-\rho_3},$$

where for simplicity, we used x_1 and x_2 instead of x^1 and x^2, respectively, and $\rho_k = \rho_k(x_1, x_2)$, $k = 1, 2, 3$, with

$$\rho_1 = (x_1 - 1)^2 + (x_2 + 1)^2, \quad \rho_2 = (x_1 + 1)^2 + (x_2 + 1)^2, \quad \rho_3 = x_1^2 + (x_2 - 1)^2.$$

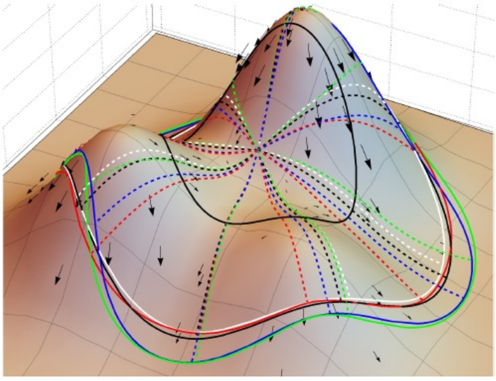
According to [13], the gravitational wind acting on $\mathfrak{G}_3$ is now

$$\mathbf{G}^T = -\frac{\bar{g}}{q+1}\left(f_{x_1}\frac{\partial}{\partial x_1} + f_{x_2}\frac{\partial}{\partial x_2}\right), \quad ||\mathbf{G}^T||_h = \bar{g}\sqrt{\frac{q}{q+1}}, \text{ with} \tag{9.55}$$

$$q = \frac{1}{4}\sum_{k=1}^{3}(k+1)^2\rho_k e^{-2\rho_k} + \frac{3}{2}(\rho_1+\rho_2-4)e^{-(\rho_1+\rho_2)} + 3(\rho_2+\rho_3-5)e^{-(\rho_2+\rho_3)} + 2(\rho_1+\rho_3-5)e^{-(\rho_1+\rho_3)}.$$

Let us denote the maximum value of the function $\mathcal{A}(x_1, x_2) = \sqrt{\frac{q}{q+1}}$ by m, considering, for example, $x_1, x_2 \in [-3, 3]$ ($m = \max\limits_{x_1,x_2\in[-3,3]} \mathcal{A}(x_1, x_2)$). Making use of a mathematical soft, an approximate value for m is 0.653 which is achieved at $(x_1, x_2) \approx (0.652, 1.272)$. Thus, for $x_1, x_2 \in [-3, 3]$, $||\mathbf{G}^T||_h \leq \max\limits_{x_1,x_2\in[-3,3]} ||\mathbf{G}^T||_h \approx 0.653\bar{g}$. The rescaled magnitude of the acceleration of gravity $\bar{g}$ needs handling with greater care to ensure that the geodesics will be indeed optimal in the sense of time. According to Theorem 9.1.1, the indicatrix of the $(\eta, \tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$ on the entire triple Gaussian bell-shaped hillside $\mathfrak{G}_3$, with $x_1, x_2 \in [-3, 3]$, is strongly convex for any $(\eta, \tilde{\eta}) \in \mathcal{S}$ if and only if $\bar{g} < \frac{1}{2\cdot\approx 0.653} \approx 0.766$. Nevertheless, by using the general condition $||\mathbf{G}^T||_h < \tilde{b}_0$, where $\tilde{b}_0$ is defined in (9.33), it is immediate to verify the following results.

Lemma 9.3.9. *The indicatrix of the $(\eta, \tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$ is strongly convex on the entire surface $\mathfrak{G}_3$, with $x_1, x_2 \in [-3, 3]$ and $m = \max\limits_{x_1,x_2\in[-3,3]} \mathcal{A}(x_1, x_2)$, if and only if $\bar{g} < \delta_2(\eta, \tilde{\eta})$, where* $\delta_2(\eta, \tilde{\eta}) = \begin{cases} \frac{1}{m(1-\tilde{\eta})}, & \text{if} \quad (\eta, \tilde{\eta}) \in \mathcal{D}_1 \cup \mathcal{D}_2 \\ \frac{1}{2m|\eta-\tilde{\eta}|}, & \text{if} \quad (\eta, \tilde{\eta}) \in \mathcal{D}_3 \cup \mathcal{D}_4 \end{cases}.$



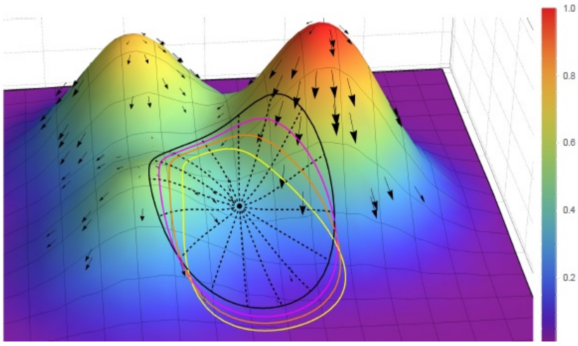


Figure 9.5: Left: On $\mathfrak{G}_3$ the time fronts centered at $(0,0)$ for the cases: MAT (green), ZNP (blue), RIEM (white), SLIPPERY (with $\eta = 0.7$, magenta), S-CROSS (with $\tilde{\eta} = 0.8$, yellow), CROSS (red) and $(0.7, 0.8)$-slope (black), as well as the related time geodesics (dashed colours, respectively), where $t = 1$ (top left) and $t = 2$ (bottom left; in addition, $(0.7, 0.8)$-case for $t = 1$); $\bar{g} = 0.76$. The time geodesics are drawn with a step of $\Delta\theta = \pi/8$ (16 time geodesics) for $t = 1$ and $\Delta\theta = \pi/4$ (8 time geodesics) for $t = 2$. The action of the gravitational wind is indicated by black arrows. Right: The evolution of the unit time front on the slippery slope of the surface $\mathfrak{G}_3$ with respect to variable force of gravitational wind (due to changing the rescaled acceleration of gravity $\bar{g}$), where $\bar{g} \in \{0.76$ (black), 3 (magenta), 5 (orange), 7.65 (yellow)$\}$. The initial point is $(1, 0)$ and the traction coefficients are fixed, i.e. $\eta = 0.7$ and $\tilde{\eta} = 0.8$. The corresponding time geodesics in the initial setting ($\bar{g} = 0.76$) are presented in dashed black and drawn with a step of $\Delta\theta = \pi/8$ (16 paths).

Now, we can write the $\tilde{F}_{\eta\tilde{\eta}}$-geodesic equations which correspond to $\mathfrak{G}_3$. According to Theorem 9.1.2 and Lemma 9.3.8, the time geodesics $\gamma(t) = (x_1(t), x_2(t))$ on the $(\eta, \tilde{\eta})$-slippery slope of the surface $\mathfrak{G}_3$ are provided by the solutions of the ODE system

$$\ddot{x}_i + f_{x_i} r_{00} + \frac{2\dot{x}_i}{\alpha}\left[\tilde{\Theta}(r_{00} + 2\alpha^2 \tilde{R} r) + \alpha\tilde{\Omega} r_0\right] + \frac{2 f_{x_i}}{q+1}\left[\tilde{\Psi}(r_{00} + 2\alpha^2 \tilde{R} r) + \alpha\tilde{\Pi} r_0\right] - 2\alpha^2 \tilde{R} r^i = 0, \tag{9.56}$$

$i = 1, 2$, where $\tilde{\Theta}$, $\tilde{R}$, $\tilde{\Omega}$, $\tilde{\Pi}$ and $\tilde{\Psi}$ are given by (9.3), q by (9.55), r^i, r, r_0 and r_{00} by (7.39) with x_1 and x_2 instead of x^1 and x^2, respectively, and everywhere in (9.56), $x_1 = x_1(t)$, $x_2 = x_2(t)$.

In order to compare all types of the slippery slopes on $\mathfrak{G}_3$ with $x_1, x_2 \in [-3, 3]$, the strong convexity conditions for the most restrictive cases, i.e. CROSS and MAT require that $\bar{g} < 0.766$. For example, we consider time geodesics and time fronts for $t \in \{1, 2\}$ for the case $\eta = 0.7$ and $\tilde{\eta} = 0.8$ on the surface $\mathfrak{G}_3$, where $\bar{g} = 0.76$ which corresponds to $||\mathbf{G}^T||_h < 0.5$. The graphical outcome is presented in Figure 9.5, left-hand side. Nevertheless, the strong convexity condition implies $||\mathbf{G}^T||_h < 5$ for the $(0.7, 0.8)$-slope. Thus, the maximum value of the rescaled gravitational acceleration can be relaxed, that is, $\bar{g} < \frac{5}{\approx 0.653} \approx 7.658$. Furthermore, the effect of a variable gravitational wind force by changing the rescaled acceleration of gravity $\bar{g}$ on behaviour of the unit time front, is pointed out on the slippery slope of $\mathfrak{G}$ in Figure 9.5, right-hand side, where the initial point is located now on the hillside, i.e. $(1, 0)$ and both traction coefficients are fixed, i.e. $\eta = 0.7$ and $\tilde{\eta} = 0.8$. The related unit time fronts are presented for $\bar{g} \in \{0.76$ (black), 3 (magenta), 5 (orange), 7.65 (yellow)$\}$.

(B-ii) The evolution and development plans for career development

Based on the results presented in **(B-i)** and also the ones obtained so far by the author, in this chapter we attempt to outline a few future research directions and career perspectives which we aim to develop.

(B-ii).1 Future research directions in complex Finsler geometry

We compile a list of the main problems we plan to focus on in the near future in the complex Finsler topic.

The study of the complex Landsberg spaces is yet to be exhausted. Lots of characterizations for both generalized Berwald and complex Landsberg spaces were presented in Chapter 2. However, some of the theoretical results are not sufficiently supported by examples. So far, it is known that every Kähler or Kähler-Berwald metric is necessarily a complex Landsberg metric, but whether there exists a complex Landsberg metric (non-pure Hermitian), which is neither generalized Berwald nor Kähler, is an open problem which can be called by analogy with the real case a *unicorn problem.* To the best of our knowledge, although a few geometers [81, 82, 101, 134, 145, 146] mentioned our results from [26], a solution for this complex version of the unicorn problem has not yet been found.

In the context of Chapter 2, as we already pointed out at the end of Section 2.2, it makes sense to also define and investigate a new class of complex spaces, for example *weak Landsberg spaces*, which hold the following relation between the horizontal coefficients of Rund and Berwald connections

$$\overset{c}{L}{}^i_{jk}\eta^j = G^i_{jk}\eta^j.$$

This new class of complex Finsler spaces generalizes the complex Landsberg spaces and moreover, it can be exemplified by the Wrona metric, given explicitly in (2.3), which is neither complex Berwald nor G-Landsberg.

The study of the projectively related complex Finsler metrics presented in Chapters 3 and 4 can be further extended in at least two directions: complex Finsler metrizability and projective metrizability, drawing on ideas from the real topics [59, 60, 57].

Let us consider a complex spray S (i.e. $S = \eta^k \frac{\partial}{\partial z^k} - 2G^k(z,\eta)\frac{\partial}{\partial \eta^k}$, with the coefficients $G^k(z,\eta)$) which does not depend on the fundamental function of a complex Finsler space (M, F). A regular curve $c : [0,1] \to M$, $c(t) = (z^i(t))$, $i = \overline{1,n}$, is called *geodesic* for S, if it is a solution of the system of second order ordinary differential equations (SODE),

$$\frac{d^2z^i}{dt^2}+2G^i(z,\frac{dz}{dt})=0,\ i=\overline{1,n},$$

with $\frac{dz^i}{dt}=\eta^i$ (see [116]). Notions of Finsler metrizabitity can also be introduced in the complex Finsler topic. Namely,

Definition *The complex spray S is complex Finsler metrizable if there exists a complex Finsler function F which satisfies $S(\bar{\eta}_k)=0$, where $\bar{\eta}_k=\frac{\partial L}{\partial \eta^k}$. Moreover, S is weakly Kähler Finsler metrizable if it is Finsler metrizable and $S(\eta_k)=\frac{\partial L}{\partial z^k}$.*

We note that the condition $S(\eta_k)=\frac{\partial L}{\partial z^k}$ is equivalent to the weakly Kähler condition for F (i.e. $\theta^{*i}=2g^{\bar{j}i}\overset{c}{\delta}_{\bar{j}}L=0,\ F^2=L$). Also, it is worth mentioning that if the spray S is weakly Kähler Finsler metrizable its geodesics are solutions of the Euler-Lagrange equations with respect to L (see [125, 116]) and thus, it is the corresponding spray of the weakly Kähler Finsler metric F. Moreover, the weakly Kähler Finsler metrizability problem can also be viewed as an inverse problem of the calculus of variation on complex manifolds restricted to weakly Kähler Finsler metrics L. More precisely, this is to find the necessary and sufficient conditions (of Helmholtz type) for the existence of two multiplier matrices $(g_{i\bar{j}}(z,\frac{dz}{dt}))$ and $(g_{ij}(z,\frac{dz}{dt}))$ such that

$$g_{i\bar{j}}(z,\frac{dz}{dt})\left(\frac{d^2\bar{z}^j}{dt^2}+2G^{\bar{j}}(z,\frac{dz}{dt})\right)+g_{ij}(z,\frac{dz}{dt})\left(\frac{d^2z^j}{dt^2}+2G^j(z,\frac{dz}{dt})\right)=\frac{d}{dt}\left(\frac{\partial L}{\partial \eta^i}\right)-\frac{\partial L}{\partial z^i},$$

$i=\overline{1,n}$, for some complex Finsler functions F.

The notion of projective metrizability appears naturally in the context of Chapters 3-4 and the above discussion. Under assumption that the coefficients G^k of the complex spray S are $(2,0)$-homogeneous and by an orientation preserving reparametrization of (SODE) (i.e. $t=t(s)$ with $\frac{dt}{ds}>0$) such that $c(s)=c(t(s))$ is a geodesic for another $(2,0)$-homogeneous complex spray $\tilde{S}$ with the coefficients $\tilde{G}^k$, we say that the homogeneous complex sprays S and $\tilde{S}$ are *projectively related.* This is equivalent to the existence of a $(1,0)$-homogeneous function $\mathcal{P}(z,\eta)$ on $T'M$ such that

$$\tilde{G}^k=G^k+\mathcal{P}(z,\eta)\eta^k.$$

Therefore, a more general geometry of the projectively related complex Finsler sprays can be developed. In particular, we say that a complex homogeneus spray is *projective Finsler metrizable* if it is projectively related to a weakly Kähler Finsler metrizable spray.

Inspired by a question of Z. Shen in [128, p. 184], another problem related to Chapter 4 that still can be extended is whether it is possible for two projectively related complex Finsler metrics to have the same $h\bar{h}$-curvature tensor. As we proved in Theorem 4.2.20, for Kähler-Berwald spaces (non-pure Hermitian) with vanishing holomorphic curvature, the answer is positive. Thus, it is natural to find an answer for non Kähler-Berwald spaces.

The Zermelo navigation problem, presented in Chapter 5, on the imaginary "sea" given by a pure Hermitian manifold (M,h) under action of a vector field (weak wind) W (i.e. $W=W^j\frac{\partial}{\partial z^j}$ and $||W||_h<1$) is not even close to being finished. It could be interesting to consider W-Zermelo deformation when W is a gradient vector field (i.e. $W=h^{\bar{m}i}\frac{\partial \omega}{\partial \bar{z}^m}\frac{\partial}{\partial z^j}$, where $\omega:M\to\mathbb{R}$ is a smooth real valued function on M) and to study the behaviour of some properties of a Hermitian metric h, e.g. Kähler property and the holomorphic sectional curvature, by the proposed Zermelo deformation. Also, bearing in mind Matsumoto's slope-of-a-mountain problem, a Hermitian approach could be tried here. For example, we set a pure Hermitian manifold (M, h) as an imaginary slope of a mountain with a gradient vector field $W = h^{\bar{m}i}\frac{\partial \omega}{\partial \bar{z}^m}\frac{\partial}{\partial z^i}$. Let the vector field $u \in T'_zM$ be the self-velocity, under assumption that $||u||_h = 1$ as it is usually set up in the standard formulation of the Zermelo navigation [45]. Under the effect of the active wind $\text{Proj}_u W$, the resultant velocity is $v = u + \text{Proj}_u W$ and the background pure Hermitian metric h is deformed into an $\mathbb{R}$-complex Finsler metric of Matsumoto type because of its real homogeneity. This is $F(z, \eta) = \frac{\alpha^2}{\alpha - \beta}$, where $\alpha^2 = h_{i\bar{j}}\eta^i\bar{\eta}^j$ and $\beta = -\text{Re } h(\eta, \bar{W})$.

(B-ii).2 Future research directions related to navigation problems

The general model presented in Chapter 9, which led to the navigation problems $\mathcal{P}_{\eta,\tilde{\eta}}$, covering the whole square $\tilde{\mathcal{S}}$ (Figure 9.1) and making close links between Matsumoto's slope-of-a-mountain problem and Zermelo's navigation problem under a gravitational wind, is currently handled. This is only the foundation for the powerful tool represented by $(\eta, \tilde{\eta})$-slope metrics that provide a big family of general (α, β)-metrics for which the study of their geometric properties is of particular interest (the flag curvature, Ricci curvature, the projective flatness, Einstein conditions, Douglas conditions, etc.). It seems reasonable to pursue several lines of investigation:

- Since the gravitational wind $\mathbf{G}^T$ is a gradient vector field, the differential 1-form β, defined in (7.14), is closed and thus each $(\eta, \tilde{\eta})$-slope metric becomes a candidate to be a Douglas metric on an n-dimensional manifold with $n \geq 3$ (see [140, Lemma 4.1]).

- Taking into consideration the paper [124, Theorem 2], an interesting study of the geodesics of the Finsler spaces with $(\eta, \tilde{\eta})$-slope metrics can be developed when the gravitational wind $\mathbf{G}^T$ is an infinitesimal homothety, this is $\mathcal{L}_{\mathbf{G}^T} h = \sigma h$, where σ is a constant. Also, in this case it could be interesting to see if it is possible to obtain a classification of $(\eta, \tilde{\eta})$-slope metrics of constant flag curvature.

- In the more general case when $\mathcal{L}_{\mathbf{G}^T} h = \sigma(x) h$ (i.e. $\mathbf{G}^T$ is conformal to h or β is conformal with respect to α), an interesting problem is to study if $(\eta, \tilde{\eta})$-slope metrics exist that are projectively flat or projectively related to α [155].

- We remark that the study of the navigation problems $\mathcal{P}_{\eta,\tilde{\eta}}$, described separately and particularly in Chapters 7 and 8, and then unified in Chapter 9, can still be expanded by assuming that the slippery slope is non-uniform. This means that either only one or both traction coefficients (cross-traction coefficient $\eta \in [0, 1]$ and along-traction coefficient $\tilde{\eta} \in [0, 1]$) could depend on the position $x \in M$, namely $\eta = \eta(x)$, $\tilde{\eta} = \tilde{\eta}(x) \in [0, 1]$. By varying one or both traction coefficients, the resultant metrics will be more extensive than the general (α, β)-metrics, namely $\tilde{F}_{\eta\tilde{\eta}}(x, y) = \alpha\tilde{\phi}_{\eta\tilde{\eta}}(||G^T||_h^2, s, \eta(x))$ or $\tilde{F}_{\eta\tilde{\eta}}(x, y) = \alpha\tilde{\phi}_{\eta\tilde{\eta}}(||G^T||_h^2, s, \tilde{\eta}(x))$ or $\tilde{F}_{\eta\tilde{\eta}}(x, y) = \alpha\tilde{\phi}_{\eta\tilde{\eta}}(||G^T||_h^2, s, \eta(x), \tilde{\eta}(x))$, because of the fact that $\tilde{\phi}_{\eta\tilde{\eta}}$ depends in addition on a third variable or on two more variables.

- Another potential extension of the navigation problems $\mathcal{P}_{\eta,\tilde{\eta}}$ may occur if we consider a varying self-speed $||u||_h$ of a craft on a slippery slope (M, h), this is $||u||_h = f(x)$, where f is a smooth function on M and $f(x) \in (0, 1]$, for any $x \in M$ (see [93]).

An interesting generalization of Matsumoto's slope-of-a-mountain problem, which we plan to develop in the future, is *Matsumoto's slope-of-a-mountain problem with wind*, mentioning that the Matsumoto metric has also been applied to a geometric description of the wildfire spread structure [105, 88]. In [88], a model for wildfire propagation with wind and slope is approached. However, we subsequently try to point out our idea for a perspective study of Matsumoto's slope-of-a-mountain problem with wind and also to answer the question formulated in [44, p. 202] regarding the Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$ in dimension two. Namely,

> *Our discussion also raises a tantalising question: if the wind were blowing on the slope of a mountain, would the indicatrix of the resulting F be a rigid translate of the limaçon?*

We consider the navigation data (F, W) on the Finsler manifold (M, F), where $F = \frac{\alpha^2}{\alpha-\bar{g}\beta}$ is restricted to $||G^T||_h < \frac{1}{2}$ (see our notations (7.14)) and the vector field W, which represents the wind in the sense of Zermelo's navigation. If we apply Proposition 6.1.1, we can provide a new Finsler metric, as well as the necessary and sufficient conditions for the strong convexity of its indicatrix, as the unique positive solution $\tilde{F}$ of the equation

$$F(x, y - \tilde{F}(x, y)W) = \tilde{F}(x, y), \qquad \text{(B.1)}$$

for any $(x, y) \in TM_0$, because of $F(x, -W) < 1$.

According to [61, p. 10 and Proposition 2.14], we note that the addition of the wind W, blowing in arbitrary directions, generates a rigid translation to the strongly convex indicatrix I_F provided by $v = u + \mathbf{G}_{MAT}$ (see the first step in Section 7.2.1 with $\eta = 1$). Thus, also in dimension two, the convex limaçon (i.e. it holds $||G^T||_h < \frac{1}{2}$) is only rigidly translated. Moreover, the condition $F(x, -W) < 1$ assures that for any $x \in M$, $y = 0$ belongs to the region bounded by the obtained indicatrix $I_{\tilde{F}}$ and is essential for the uniqueness of the solution of the equation (B.1) (see [61, p. 10 and Proposition 2.14]). Further on, the investigation of the restriction $F(x, -W) < 1$, for $F = \frac{\alpha^2}{\alpha-\bar{g}\beta}$ with $||G^T||_h < \frac{1}{2}$, leads to the following necessary and sufficient conditions:

$$h(W, G^T) < ||W||_h(1 - ||W||_h) \quad \text{and} \quad ||G^T||_h < \frac{1}{2} \qquad \text{(B.2)}$$

for the strong convexity of the indicatrix $I_{\tilde{F}}$. Following (B.1), the deformation of the Matsumoto metric $F = \frac{\alpha^2}{\alpha-\bar{g}\beta}$ by the wind W restricted to (B.2) is the Finsler metric $\tilde{F}$ which satisfies

$$\tilde{F}\left(\sqrt{||y||_h^2 - 2h(y, W)\tilde{F} + ||W||_h^2\tilde{F}^2} + h(y, \mathbf{G}^T) - h(W, \mathbf{G}^T)\tilde{F}\right) = ||y||_h^2 - 2h(y, W)\tilde{F} + ||W||_h^2\tilde{F}^2,$$

where $\tilde{F}$ is evaluated at (x, y). The last relation is the main ingredient to arrive at the spray coefficients corresponding to the Finsler metric $\tilde{F}$ and then to write the ODE system (6.2) which provides $\tilde{F}$-geodesics. Since along any regular piecewise C^∞-curve γ, parametrized by time (i.e. the time in which a craft or a vehicle goes along γ) that represents a trajectory in

Zermelo's problem, it is satisfied the equality $\tilde{F}(\gamma(t),\dot{\gamma}(t)) = 1$, by $\tilde{F}$-geodesics one can get the time-minimizing paths when the wind is blowing on the slope of a mountain. Beyond the above meaning of the Finsler metric $\tilde{F}$ provided by the last equation, an important direction is also the study of different geometric properties of $\tilde{F}$ (the flag curvature, Ricci curvature, the projective flatness, Einstein conditions, Douglas conditions, etc.).

Bearing in mind the aforementioned perspective that Matsumoto's slope-of-a-mountain problem with wind is solvable, a natural question that arises is how to navigate on a slippery slope of a mountain in the presence of a wind in order to come from one point to another point in the shortest time? The answer of this problem represents a significant future direction of study that has to include the solution of the Matsumoto's slope-of-a-mountain problem with wind. More precisely, let us consider for example the slippery slope of a mountain represented by the n-dimensional Riemannian manifold (M,h), $n>1$, with the gravitational wind $\mathbf{G}^T$ and the cross-traction coefficient $\eta \in [0,1]$. The time-minimal paths on (M,h) in the presence of an active wind $\mathbf{G}_\eta$ defined by (7.3) and under the influence of a arbitrary wind W (i.e. with the equation of motion $v = u + \mathbf{G}_\eta + W$) are the geodesics of the Finsler metric $\mathcal{F}_\eta$ which satisfies

$$\begin{aligned} &\mathcal{F}_\eta \left\{(||W||_h^2+\Omega_1)\mathcal{F}_\eta^2 - 2[h(y,W)+(1-\eta)h(y,\mathbf{G}^T)]\mathcal{F}_\eta + ||y||_h^2\right\}^{1/2} \\ &= (||W||_h^2+\Omega_2)\mathcal{F}_\eta^2 - [2h(y,W)+(2-\eta)h(y,\mathbf{G}^T)]\mathcal{F}_\eta + ||y||_h^2, \end{aligned} \tag{B.3}$$

where $\mathcal{F}_\eta$ is evaluated at (x,y), under the restrictions

$$||W||_h^2 + \Omega_1 < \sqrt{||W||_h^2+\Omega_2} \text{ and } ||\mathbf{G}^T||_h < \tilde{b}_0, \tag{B.4}$$

with $\Omega_1 = (1-\eta)[2h(W,\mathbf{G}^T) + (1-\eta)||\mathbf{G}^T||_h^2]$, $\Omega_2 = (2-\eta)h(W,\mathbf{G}^T) + (1-\eta)||\mathbf{G}^T||_h^2$ and either $\tilde{b}_0 = 1$ if $\eta \in \left[0,\frac{1}{2}\right]$ or $\tilde{b}_0 = \frac{1}{2\eta}$ if $\eta \in \left(\frac{1}{2},1\right]$. In particular, if $\eta = 1$ in (B.3) and (B.4), one can extract the Finsler metric $\tilde{F}$ which solves the Matsumoto's slope-of-a-mountain problem with wind.

(B-ii).3 Further perspectives

A brief list of the author's background is presented below:

Research publications. After obtaining a Ph.D. degree in Mathematics (2005), among the research papers that have been published or accepted for publication, worth mentioning are:

- 8 published in the Q1[7] journals (such as: 2 in Nonlinear Analysis - Theory Methods and Applications [10, 11], 2 in Nonlinear Analysis - Real World Applications [12, 25], 1 in Journal of the Franklin Institute-Engineering and Applied Mathematics [17], 1 in Journal of Optimization Theory and Applications [16], 1 in Annual Reviews in Control [19], 1 in The Journal of Navigation [18]),
- 10 in the Q2 journals (such as: 3 in Journal of Geometry and Physics [23, 26, 31], 3 in Results in Mathematics [9, 15, 21], 2 in Differential Geometry and its Applications [14, 24], 1 in Periodica Mathematica Hungarica [20], 1 in Acta Mathematica Scientia [28]),

[7] According to the ranking AIS lists, the last five editions (2020-2024); Q1-red area, Q2-yellow area; https://uefiscdi.gov.ro/scientometrie-reviste

- 1 accepted for publication in The Journal of Geometric Analysis (Q1 journal), [13].

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Didactic activities. During the last 10 years - advisor for 34 bachelor or master theses in Differential geometry and Linear algebra.

The overall aim is to extend and enhance the research significantly in the aforementioned directions as well as to explore new avenues that may contribute to confering a higher academic position to the author of this thesis. The results outlined above provide supporting evidence that this objective is both realistic and achievable.

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