Contributions to Complex Finsler Geometry
Models for Optimal Navigation under Gravity - A Finsler Approach

Habilitation Thesis

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Main objectives

1. To contribute to the development of complex Finsler geometry, addressing some aspects related to
   - complex Landsberg spaces,
   - projectivity,
   - holomorphic curvature,
   - deformation.
2. To extend Matsumoto's slope-of-a-mountain problem through a general model of time-optimal navigation based on Riemann-Finsler geometry, handling
   - one-parameter models,
   - two-parameter model.

General structure

1. Part I. Different aspects of complex Finsler geometry
    - Chapter 1: Rudiments of complex Finsler geometry
    - Chapter 2: On complex Landsberg spaces
    - Chapter 3: Projectivities in complex Finsler geometry
    - Chapter 4: Projective invariants of a complex Finsler space
    - Chapter 5: Zermelo's deformation of Hermitian metrics
2. Part II. Extensions of Matsumoto's slope-of-a-mountain problem
    - Chapter 6: Rudiments of real Finsler geometry
    - Chapter 7: Time geodesics on a slippery slope under gravitational wind
    - Chapter 8: The slope-of-a-mountain problem in a cross gravitational wind
    - Chapter 9: A general model for time-minimizing navigation on a mountain slope under gravity
3. Further research: directions & perspectives

General view

Complex Finsler geometry is extremely beautiful (S. S. Chern, 1996)

Aim: to introduce general themes from real Finsler geometry into complex Finsler geometry.

Approached problems:

- Detailed study of the complex Landsberg and generalized Berwald spaces; particular cases of complex Landsberg spaces (Ch. 2);
- Projectively related complex Finsler metrics; complex versions of Rapcsák's theorem; a complex Finsler solution of Hilbert's fourth problem (Ch. 3);
- Exploration of the projective curvature invariants of Douglas and Weyl type; complex Finsler spaces of constant holomorphic curvature; complex Douglas spaces (Ch. 4);
- Zermelo's navigation problem on a Hermitian manifold; how some properties of a Hermitian metric are affected by the Zermelo deformation under action of some special winds (Ch. 5).

1.1 Complex Finsler spaces

Main refs.: M. Abate & G. Patrizio (1994), G. Munteanu (2004)

- M - n-dimensional complex manifold;
- $z = (z^k) \in M,\ k = \overline{1,n}$ - the complex coordinates in a local chart;
- $T_{\mathbb{R}}M \rightsquigarrow T_{\mathbb{C}}M = T'M \oplus T''M$, where $T'M$ - holomorphic tangent bundle, $\pi : T'M \to M$; $T''M$ - antiholomorphic tangent bundle;
- $T'M$ is a complex manifold $\rightsquigarrow$ the local coordinates $u = (z^k, \eta^k)$.

Definition

(M, F) is a **complex Finsler space**, where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:
i) $L = F^2$ is smooth on $\widetilde{T'M} = T'M \backslash \{0\}$;
ii) $F(z, \eta) \geq 0$, for all $F(z, \eta) \in T'M$; the equality holds iff $\eta = 0$;
iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, $\forall \lambda \in \mathbb{C},\ \lambda \neq 0$;
iv) the Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$ is positive definite, where $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ the fundamental metric tensor.

1.1 Complex Finsler spaces

- $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$;
- $VT'M = \ker \pi_* \subset T'(T'M)$ - the vertical bundle; $V_uT'M \rightsquigarrow \{\frac{\partial}{\partial \eta^k}\}$;
- A complex nonlinear connection (c.n.c.) is a complex subbundle of $T'(T'M)$, such that $T'(T'M) = HT'M \oplus VT'M$;
- $H_uT'M \rightsquigarrow \{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$; $N_k^j(z,\eta)$ - the coefficients of the (c.n.c.);
- $\{\delta_k = \frac{\delta}{\delta z^k}, \dot{\partial}_k = \frac{\partial}{\partial \eta^k}\}$ - the adapted frame on $T'_u(T'M)$ of the (c.n.c.);
- By conjugation $\rightsquigarrow$ the adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$;
- $\{dz^k, \delta\eta^k = d\eta^k + N_j^k dz^j\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$ - the adapted dual frames;
- A section on $T'(T'M)$, locally expressed as $S = \eta^k \frac{\partial}{\partial z^k} - 2G^k(z,\eta)\dot{\partial}_k$ is a **complex spray**, where G^k denote the spray coefficients;
- **Chern-Finsler** (c.n.c.) with the local coefficients

$$N_j^i = g^{\overline{m}i} \frac{\partial g_{l\overline{m}}}{\partial z^j} \eta^l; \tag{1}$$

- $\rightsquigarrow \delta_k$ is w.r.t. Chern-Finsler (c.n.c.).

1.1 Complex Finsler spaces

- **Chern-Finsler connection** D: i) of $(1,0)$-type (i.e. $D_{JX}Y = JD_XY, \forall\, X$ section on $T'(T'M)$, $\forall\, Y$ vertical vector field) ii) metrical w.r.t. the Hermitian structure; Locally, $CF\Gamma = (N^i_j, L^i_{jk}, L^{\bar{\imath}}_{\bar{j}k}, C^i_{jk}, C^{\bar{\imath}}_{\bar{j}k})$, where

$$N^i_j = L^i_{lj}\eta^l, \quad L^i_{jk} = g^{\bar{l}i}\delta_k g_{j\bar{l}} = \dot{\partial}_j N^i_k, \quad C^i_{jk} = g^{\bar{l}i}\dot{\partial}_k g_{j\bar{l}}, \quad L^{\bar{\imath}}_{\bar{j}k} = C^{\bar{\imath}}_{\bar{j}k} = 0. \quad (2)$$

- **Complex Cartan tensors**: $C_{i\bar{j}k} = \dot{\partial}_k g_{i\bar{j}}$ and $C_{i\bar{j}\bar{k}} = \dot{\partial}_{\bar{k}} g_{i\bar{j}}$;
- $R^i_{j\bar{h}k} = -\delta_{\bar{h}} L^i_{jk} - (\delta_{\bar{h}} N^l_k) C^i_{jl}$ denote the $h\bar{h}$ - curvatures coefficients of D;
- **Holomorphic curvature** of the complex Finsler space (M,F) in direction η

$$\mathcal{K}_F(z,\eta) = \frac{2}{L^2} R_{\bar{r}j\bar{k}h}\bar{\eta}^r\eta^j\bar{\eta}^k\eta^h, \quad \text{where} \quad R_{\bar{r}j\bar{k}h} = R^i_{j\bar{k}h} g_{i\bar{r}}. \quad (3)$$

- (M,F) is **strongly Kähler** iff $T^i_{jk} = 0$;
- (M,F) is **Kähler** iff $T^i_{jk}\eta^j = 0$;
- (M,F) is **weakly Kähler** iff $g_{i\bar{l}} T^i_{jk}\eta^j\bar{\eta}^l = 0$, where $T^i_{jk} = L^i_{jk} - L^i_{kj}$;
- strongly Kähler $\equiv$ Kähler [B. Chen, Y. Shen, Chin. Ann. Math. 2009];
- weakly Kähler $\equiv$ Kähler for **pure Hermitian metrics**, i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$;

1.2 Connections on a complex Finsler space

- Chern-Finsler (c.n.c.) determines a complex spray with the local coefficients

$$2G^i = N^i_j \eta^j; \tag{4}$$

- G^i induce a (c.n.c.) by $\overset{c}{N^i_j} = \dot{\partial}_j G^i$ called **canonical (c.n.c)**;
- $\overset{c}{\delta_k}$ is related to canonical (c.n.c.), i.e. $\overset{c}{\delta_k} = \frac{\partial}{\partial z^k} - \overset{c}{N^j_k} \dot{\partial}_j$;
- W.r.t. the canonical (c.n.c.) we consider two linear connections:

i) of Berwald type $B\Gamma = (\overset{c}{N^i_j}, G^i_{jk}, G^i_{j\bar{k}}, 0, 0)$ having the connection form

$$\omega^i_j(z,\eta) = G^i_{jk} dz^k + G^i_{j\bar{k}} d\bar{z}^k, \tag{5}$$

where $G^i_{jk} = \dot{\partial}_k \overset{c}{N^i_j} = G^i_{kj}$ and $G^i_{j\bar{k}} = \dot{\partial}_{\bar{k}} \overset{c}{N^i_j}$;

ii) of Rund type $R\Gamma = (\overset{c}{N^i_j}, \overset{c}{L^i_{jk}}, \overset{c}{L^i_{j\bar{k}}}, 0, 0)$, where

$\overset{c}{L^i_{jk}} = \frac{1}{2} g^{\bar{l}i} (\overset{c}{\delta_k} g_{j\bar{l}} + \overset{c}{\delta_j} g_{k\bar{l}})$ and $\overset{c}{L^i_{j\bar{k}}} = \frac{1}{2} g^{\bar{l}i} (\overset{c}{\delta_{\bar{k}}} g_{j\bar{l}} - \overset{c}{\delta_{\bar{l}}} g_{j\bar{k}})$.

1.2 Connections on a complex Finsler space

- $R\Gamma$ is only h-metrical and $B\Gamma$ is neither h- nor v-metrical;
- $2G^i = N^i_j\eta^j = \overset{c}{N}{}^i_j\eta^j = G^i_{jk}\eta^j\eta^k$ and $\overset{c}{\delta}_j = \delta_j - (\overset{c}{N}{}^k_j - N^k_j)\dot{\partial}_k$;
- If (M, F) is Kähler $\Rightarrow N^k_j = \overset{c}{N}{}^i_j$, $\overset{c}{\delta}_k = \delta_k$, $\overset{c}{L}{}^i_{h\bar{j}} = 0$, $L^i_{jk} = \overset{c}{L}{}^i_{jk} = G^i_{jk}$.

Lemma [N.A., G. Munteanu, J. Geom. Phys. 2012]

For any complex Finsler space (M, F), the following statements hold:

i) $G^i_{j\bar{k}}\bar{\eta}^k = 0$;

ii) $g_{l\bar{r}\overset{B}{|}h} + g_{h\bar{r}\overset{B}{|}l} + G^{\bar{m}}_{\bar{r}h}g_{l\bar{m}} + G^{\bar{m}}_{\bar{r}l}g_{h\bar{m}} = -C_{l\bar{r}h\overset{B}{|}0}$;

iii) $2(\dot{\partial}_{\bar{h}}G^i)g_{i\bar{r}} = C_{0\bar{r}\bar{h}\overset{B}{|}0} = C_{0\bar{r}\bar{h}|0}$;

iv) $C_{i\bar{j}h\overset{B}{|}k} = \dot{\partial}_h(g_{i\bar{j}\overset{B}{|}k}) + (\dot{\partial}_h G^l_{ik})g_{l\bar{j}} + (\dot{\partial}_h G^{\bar{m}}_{\bar{j}k})g_{i\bar{m}}$;

v) $C_{i\bar{r}\bar{h}\overset{B}{|}k} = \dot{\partial}_{\bar{h}}(g_{i\bar{j}\overset{B}{|}k}) + (\dot{\partial}_{\bar{h}}G^l_{ik})g_{l\bar{j}} + (\dot{\partial}_{\bar{h}}G^{\bar{m}}_{\bar{j}k})g_{i\bar{m}} + G^l_{k\bar{h}}C_{i\bar{j}l} - G^{\bar{m}}_{\bar{h}k}C_{i\bar{j}\bar{m}}$,

where $\overset{B}{|}$ is h-covariant derivative with respect to $B\Gamma$.

2.2 From complex Landsberg to generalized Berwald spaces

N. Aldea, G. Munteanu, On complex Landsberg and Berwald spaces. *J. Geom. Phys.*, 62(2) (2012), 368-380.

Definition

Let (M,F) be an n-dimensional complex Finsler space. (M,F) is called a complex Landsberg space if $G^i_{jk}=\overset{c}{L}{}^i_{jk}$.

- Kähler spaces offer an asset family of complex Landsberg spaces.

Theorem

Let (M,F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:
i) (M,F) *is a complex Landsberg space;*
ii) $C_{l\bar{r}h\,\overset{B}{|}\,0}=0$;
iii) $2(\dot{\partial}_h G^i_{jk})g_{i\bar{r}}-G^{\bar{m}}_{\bar{r}k}C_{j\bar{m}h}-G^{\bar{m}}_{\bar{r}j}C_{k\bar{m}h}=C_{j\bar{r}h\,\overset{B}{|}\,k}+C_{k\bar{r}h\,\overset{B}{|}\,j}$;
iv) $g_{i\bar{j}\,\overset{B}{|}\,k}=(\overset{c}{L}{}^{\bar{m}}_{\bar{j}k}-G^{\bar{m}}_{\bar{j}k})g_{i\bar{m}}$.

2.2 From complex Landsberg to generalized Berwald spaces

Definition

Let (M,F) be an n-dimensional complex Finsler space. (M,F) is called a G-Landsberg space if it is Landsberg and the spray coefficients G^i are holomorphic with respect to η, i.e. $\dot{\partial}_{\bar{k}}G^i = 0$.

Theorem

Let (M,F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:

i) (M,F) *is a* G*-Landsberg space;*

ii) $G^i_{jk} = \overset{c}{L}{}^i_{jk}(z)$;

iii) $C_{l\bar{r}h \overset{B}{|} 0} = 0$ *and* $C_{j\bar{0}h \overset{B}{|} \bar{0}} = 0$;

iv) $g_{i\bar{j} \overset{B}{|} k} = \overset{c}{L}{}^{\bar{m}}_{\bar{j}k} g_{i\bar{m}}$ *and* $\dot{\partial}_{\bar{h}}G^i = 0$.

v) $C_{j\bar{r}h \overset{B}{|} k} + C_{k\bar{r}h \overset{B}{|} j} = 0$ *and* $C_{r\bar{l}h \overset{B}{|} \bar{k}} + C_{r\bar{k}h \overset{B}{|} \bar{l}} = 0$.

2.2 From complex Landsberg to generalized Berwald spaces

Definition

Let (M, F) be an n-dimensional complex Finsler space. (M, F) is called a strong Landsberg space if $C_{l\bar{r}h\overset{B}{|}0} = 0$ and $C_{j\bar{r}h\overset{B}{|}\bar{0}} = 0$.

Theorem

Let (M, F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:

i) (M, F) is a strong Landsberg space;

ii) $g_{l\bar{r}\overset{B}{|}s}(z)$ and $\dot{\partial}_{\bar{h}}G^i = 0$;

iii) $C_{l\bar{r}h\overset{B}{|}k} = 0$ and $\dot{\partial}_{\bar{h}}G^i = 0$;

iv) $C_{j\bar{r}h\overset{B}{|}\bar{k}} = 0$.

Definition

Let (M, F) be an n-dimensional complex Finsler space. (M, F) is called a G-Kähler space if it is Kähler and the spray coefficients G^i are holomorphic w.r.t. η.

2.2 From complex Landsberg to generalized Berwald spaces

- (M, F) is complex **Berwald** iff $L^i_{jk}(z)$ [T. Aikou, Contemp. Math., 196, 1996];
- $L^i_{jk}(z) \Leftrightarrow C_{l\bar{r}h|k} = 0 \Leftrightarrow C_{l\bar{r}h|\bar{k}} = 0.$

Theorem

(M, F) is Kähler-Berwald space if and only if it is Kähler and either $C_{l\bar{r}h|\bar{k}} = 0$ or $C_{l\bar{r}h|k} = 0$.

Theorem

Let (M, F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:
i) (M, F) is a G-Kähler space;
ii) $G^i_{j\bar{k}} = \overset{c}{L}{}^i_{j\bar{k}}$;
iii) $G^i_{jk} = L^i_{jk}(z)$;
iv) (M, F) is a Kähler-Berwald space;
v) $g_{i\bar{j}\overset{B}{|}k} = 0$ and $\dot{\partial}_{\bar{h}} G^i = 0$.

2.2 From complex Landsberg to generalized Berwald spaces

Definition

Let (M, F) be an n-dimensional complex Finsler space. (M, F) is called generalized Berwald if the horizontal coefficients G^i_{jk} of $B\Gamma$ depend only on the position z.

Theorem

Let (M, F) be an n-dimensional complex Finsler space. Then the following assertions are equivalent:
i) (M, F) is generalized Berwald;
ii) G^i are holomorphic with respect to η;
iii) $B\Gamma$ is of $(1, 0)$-type.

Corollary

If (M, F) is a complex Berwald space, then the space is generalized Berwald.

- generalized Berwald $\equiv$ weakly complex Berwald [C. Zhong, Diff. Geom. Appl. 2011].

Examples

- **Complex version of the *Wrona* metric** $\rightsquigarrow$ generalized Berwald

$$F(z,\eta) = \frac{|PQ|}{|OH|} = \frac{|\eta|^4}{|z|^2|\eta|^2 - | < z,\eta > |^2}, \tag{6}$$

with $(z,\eta) \in \Omega = \{(z,\eta) \in \mathbb{C}^n \times \mathbb{C}^n \mid z \neq \lambda\eta,\ \lambda \in \mathbb{C}\}$, where $P,\ Q \in \mathbb{C}^n$, O is the origin of $\mathbb{C}^n$, H is the projection of O on the line PQ.

$\rightsquigarrow G^i = 0$ and $\overset{c}{L}{}^i_{jk} \neq G^i_{jk} \rightsquigarrow$ (6) is a generalized Berwald metric which is neither G-Landsberg nor Berwald; it satisfies $\overset{c}{L}{}^i_{jk}\eta^j = G^i_{jk}\eta^j$.

- **Complex version of the *Antonelli-Shimada* metric** $\rightsquigarrow$ complex Berwald

$$F^2_{AS} = L_{AS}(z,w;\eta,\theta) = e^{2\sigma}\left(|\eta|^4 + |\theta|^4\right)^{\frac{1}{2}}, \text{ with } \eta,\theta \neq 0, \tag{7}$$

on a domain D from $\widetilde{T'M}$, $\dim M = 2$, such that its metric tensor is nondegenerated. The non-zero coefficients:

$$L^1_{11} = L^2_{21} = 2\frac{\partial\sigma}{\partial z} \text{ and } L^1_{12} = L^2_{22} = 2\frac{\partial\sigma}{\partial w},$$

depend only on z and w. L_{AS} is not G-Landsberg ($\overset{c}{L}{}^i_{jk} \neq G^i_{jk}$).

2.2 From complex Landsberg to generalized Berwald spaces

Theorem [N.A, G. Munteanu, Diff. Geom. Appl. 2012]

Let (M, F) be a complex Finsler space that satisfies the weakly Kähler and generalized Berwald conditions. Then (M, F) is a Kähler-Berwald space.

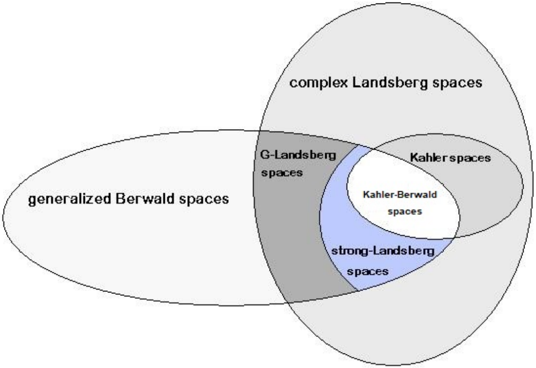


Figure 1: Inclusion diagram

3.2 Projectively related complex Finsler metrics

N. Aldea, G. Munteanu, Projectively related complex Finsler metrics. *Nonlinear Anal. Real World Appl.* 13(5) (2012), 2178-2187.

In Abate-Patrizio's sense [M. Abate, G. Patrizio 1994], a geodesic curve is provided by

$$D_{T^h+\overline{T^h}}T^h = \theta^*(T^h,\overline{T^h}), \tag{8}$$

where T^h is the horizontal lift of the tangent vector along the curve and

$$\theta^* = g^{\bar{m}k}g_{i\bar{p}}(L^{\bar{p}}_{\bar{j}\bar{m}} - L^{\bar{p}}_{\bar{m}\bar{j}})dz^i \wedge d\bar{z}^j \otimes \delta_k. \tag{9}$$

The equations of a geodesic $z = z(s)$ of (M,F), with s a real parameter, can be rewritten as

$$\frac{d^2z^i}{ds^2} + 2G^i(z(s),\frac{dz}{ds}) = \theta^{*i}(z(s),\frac{dz}{ds}), \tag{10}$$

where $z^i(s)$, $i = \overline{1,n}$, are the coordinates along of the curve $z = z(s)$ and $\theta^{*k} = 2g^{\bar{j}k}\ \overset{c}{\delta}_{\bar{j}}\ L$.

- θ^{*i}=0 iff (M,F) is weakly Kähler.
- θ^{*i} are $(1,1)$-homogeneous w.r.t. η and $\bar{\eta}$ respectively, i.e. $(\dot{\partial}_k\theta^{*i})\eta^k = \theta^{*i}$ and $(\dot{\partial}_{\bar{k}}\theta^{*i})\bar{\eta}^k = \theta^{*i}$.

3.2 Projectively related complex Finsler metrics

Let $\tilde{F}$ be another complex Finsler metric on the underlying manifold M.

Definition

The complex Finsler metrics F and $\tilde{F}$ on manifold M are called projectively related if they have the same geodesics as point sets.

Theorem

Let F and $\tilde{F}$ be complex Finsler metrics on M. Then F and $\tilde{F}$ are projectively related if and only if there is a smooth function P on $T'M$ with complex values, such that

$$\tilde{G}^i = G^i + B^i + P\eta^i, \tag{11}$$

where $B^i = \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$, $i = \overline{1,n}$.

- (11) - projective change

Corollary

Let F and $\tilde{F}$ be complex Finsler metrics on M. F and $\tilde{F}$ are projectively related if and only if there is a smooth function P on $T'M$, such that $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k\eta^i$, $B^i = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k\eta^i$ *and* $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$, *for any* $i = \overline{1,n}$.

3.2 Projectively related complex Finsler metrics $\rightsquigarrow$ Rapcsák's theorem

Theorem

Let F and $\tilde{F}$ be two projectively related complex Finsler metrics on M. Then, F is weakly Kähler if and only if $\tilde{F}$ is also weakly Kähler. In this case, the projective change is $\tilde{G}^i = G^i + P\eta^i$, where P is a $(1,0)$-homogeneous function.

Lemma

Let L and $\tilde{L}$ be complex Finsler metrics on M. The spray coefficients $\tilde{G}^i$ and G^i of the metrics L and $\tilde{L}$ satisfy

$$\tilde{G}^i = G^i + \frac{1}{2}\tilde{g}^{\bar{r}i}\left[\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l \tilde{L})\right], \; i = \overline{1,n}. \tag{12}$$

Theorem

Let L and $\tilde{L}$ be complex Finsler metrics on M. Then, L and $\tilde{L}$ are projectively related if and only if

$$\frac{1}{2}\left[\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l \tilde{L})\right] = P(\dot{\partial}_{\bar{r}}\tilde{L}) + B^i\tilde{g}_{i\bar{r}}, \; r = \overline{1,n}, \tag{13}$$

with $P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})]$.

3.2 Projectively related complex Finsler metrics $\rightsquigarrow$ Rapcsák's theorem

Theorem

Let L and $\tilde{L}$ be complex Finsler metrics on M. Then, L and $\tilde{L}$ are projectively related if and only if

$$\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}) = \frac{1}{\tilde{L}}(\delta_k \tilde{L})\eta^k(\dot{\partial}_{\bar{r}} \tilde{L}), \tag{14}$$

$$B^r = -\frac{1}{2\tilde{L}}\theta^{*l}(\dot{\partial}_l \tilde{L})\eta^r,\ r = \overline{1,n},$$

$$P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})].$$

Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k \eta^i$.

Theorem

Let L be a weakly Kähler complex Finsler metric and $\tilde{L}$ be another complex Finsler metric, both on M. Then, L and $\tilde{L}$ are projectively related if and only if $\tilde{L}$ is weakly Kähler and

$$\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}) = 2P(\dot{\partial}_{\bar{r}} \tilde{L}),\ r = \overline{1,n}, \tag{15}$$

$$P = \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k.$$

Moreover, the projective change is $\tilde{G}^i = G^i + P\eta^i$ *and* P *is* $(1,0)$*-homogeneous.*

3.2 Projectively related complex Finsler metrics ⇝ Rapcsák's theorem

Corollary

Let F be a generalized Berwald metric and $\tilde{F}$ be another complex Finsler metric, both on M. Then, F and $\tilde{F}$ are projectively related if and only if

$$\dot{\partial}_{\bar{r}}(\delta_k\tilde{F})\eta^k = \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k(\dot{\partial}_{\bar{r}}\tilde{F}),\ B^r = -\frac{1}{\tilde{F}}\theta^{*l}(\dot{\partial}_l\tilde{F})\eta^r, \tag{16}$$
$$P = \frac{1}{\tilde{F}}[(\delta_k\tilde{F})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{F})],$$

for any $r = \overline{1,n}$. Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k\eta^i$ and $\tilde{F}$ is also generalized Berwald.

Corollary

Let F be a Kähler-Berwald metric and $\tilde{F}$ be another complex Finsler metric, both on M. Then, F and $\tilde{F}$ are projectively related if and only if $\tilde{F}$ is weakly Kähler and

$$\dot{\partial}_{\bar{r}}(\delta_k\tilde{F})\eta^k = P(\dot{\partial}_{\bar{r}}\tilde{F}),\ r = \overline{1,n} \text{ and } P = \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k. \tag{17}$$

Moreover, the projective change is $\tilde{G}^i = G^i + P\eta^i$ and $\tilde{F}$ is Kähler-Berwald.

3.2 Projectively related complex Finsler metrics $\rightsquigarrow$ Hilbert's fourth problem

Theorem

Let L be the complex Euclidean metric on a domain D from $\mathbf{C}^n$ and $\tilde{L}$ be another complex Finsler metric on D. Then, L and $\tilde{L}$ are projectively related if and only if $\tilde{L}$ is weakly Kähler and

$$\tilde{G}^i = \frac{1}{2\tilde{L}}\frac{\partial \tilde{L}}{\partial z^k}\eta^k\eta^i,\ i = \overline{1,n}. \tag{18}$$

Moreover, $\tilde{L}$ is Kähler-Berwald.

- (18) $\Rightarrow \tilde{G}^i = \frac{1}{\tilde{F}}\frac{\partial \tilde{F}}{\partial z^k}\eta^k\eta^i,\ i = \overline{1,n}.$
- Examples of complex Finsler metrics which are projectively related to the complex Euclidean metric:

$$\tilde{F}^2(z,\eta) = \frac{|\eta|^2 + \varepsilon\left(|z|^2|\eta|^2 - |< z,\eta >|^2\right)}{(1+\varepsilon|z|^2)^2},\ \varepsilon < 0, \tag{19}$$

defined on the disk $\Delta_r^n = \left\{z \in \mathbf{C}^n,\ |z| < r,\ \ r = \sqrt{1/|\varepsilon|}\right\} \subset \mathbf{C}^n$.

$\rightsquigarrow \tilde{G}^i = -\frac{\varepsilon\overline{<z,\eta>}}{1+\varepsilon|z|^2}\eta^i = \frac{1}{\tilde{F}}\frac{\partial \tilde{F}}{\partial z^k}\eta^k\eta^i$;

$\rightsquigarrow$ the metrics (19) are Kähler, pure Hermitian with $\mathcal{K}_{\tilde{F}} = 4\varepsilon$;

$\rightsquigarrow$ for $\varepsilon = -1$, (19) provides the Bergman metric on the unit disk $\Delta^n = \Delta_1^n$.

4.2 Projective curvature invariants

N. Aldea, G. Munteanu, On projective invariants of the complex Finsler spaces. *Diff. Geom. Appl.*, 30(6) (2012), 562-575.

N. Aldea, G. Munteanu, On complex Douglas spaces. *J. Geom. Phys.*, 66 (2013), 80-93.

- Key tool: Berwald type connection $B\Gamma \rightsquigarrow$ the canonical (c.n.c.);
- Connection form of $B\Gamma$: $\omega^i_j(z,\eta) = G^i_{jk}dz^k + G^i_{j\bar{k}}d\bar{z}^k$;
- The structure equation: $d\omega^i_j - \omega^k_j \wedge \omega^i_k = \Omega^i_j$ with the curvature form:

$$\Omega^i_j = -\frac{1}{2}K^i_{jkh}dz^k \wedge dz^h - \frac{1}{2}K^i_{j\bar{k}\bar{h}}d\bar{z}^k \wedge d\bar{z}^h + K^i_{j\bar{h}k}dz^k \wedge d\bar{z}^h$$
$$- G^i_{jkh}dz^k \wedge \overset{c}{\delta}\eta^h - G^i_{j\bar{k}\bar{h}}d\bar{z}^k \wedge \overset{c}{\delta}\bar{\eta}^h - G^i_{j\bar{h}k}dz^k \wedge \overset{c}{\delta}\bar{\eta}^h + G^i_{j\bar{h}k}\overset{c}{\delta}\eta^k \wedge d\bar{z}^h;$$

- hh-, $\bar{h}\bar{h}$- and $h\bar{h}$- curvature tensors:

$$\begin{aligned} K^i_{jkh} &= \overset{c}{\delta}_h G^i_{jk} - \overset{c}{\delta}_k G^i_{jh} + G^l_{jk}G^i_{lh} - G^l_{jh}G^i_{lk}, \\ K^i_{j\bar{k}\bar{h}} &= \overset{c}{\delta}_{\bar{h}} G^i_{j\bar{k}} - \overset{c}{\delta}_{\bar{k}} G^i_{j\bar{h}} + G^l_{j\bar{k}}G^i_{l\bar{h}} - G^l_{j\bar{h}}G^i_{l\bar{k}}, \\ K^i_{j\bar{k}h} &= \overset{c}{\delta}_h G^i_{j\bar{k}} - \overset{c}{\delta}_{\bar{k}} G^i_{jh} + G^l_{j\bar{k}}G^i_{lh} - G^l_{jh}G^i_{l\bar{k}}. \end{aligned}$$

4.2.1 Curvature forms and Bianchi identities

- hv-, $\bar{h}\bar{v}$- and $h\bar{v}$- curvature tensors:

$$G^i_{jkh} = \dot{\partial}_h G^i_{jk}, \; G^i_{j\bar{k}\bar{h}} = \dot{\partial}_{\bar{h}} G^i_{j\bar{k}}, \; G^i_{j\bar{k}h} = \dot{\partial}_h G^i_{j\bar{k}}.$$

- Bianchi identities:

$$\dot{\partial}_r G^i_{jkh} = \dot{\partial}_h G^i_{jkr}, \; \dot{\partial}_r G^i_{j\bar{k}h} = \dot{\partial}_h G^i_{j\bar{k}r}, \; \dot{\partial}_r G^i_{j\bar{k}\bar{h}} = \dot{\partial}_{\bar{h}} G^i_{j\bar{k}r},$$
$$\dot{\partial}_{\bar{r}} G^i_{jkh} = \dot{\partial}_h G^i_{j\bar{r}k}, \; \dot{\partial}_{\bar{r}} G^i_{j\bar{k}\bar{h}} = \dot{\partial}_{\bar{h}} G^i_{j\bar{k}\bar{r}}, \; \dot{\partial}_{\bar{r}} G^i_{j\bar{h}k} = \dot{\partial}_{\bar{h}} G^i_{j\bar{r}k}.$$

- θ^{*i} hold:

$$\theta^{*i}_{kj}\eta^k = 0, \; \theta^{*i}_{k\bar{h}}\eta^k = \theta^{*i}_{\bar{h}}, \; \theta^{*i}_{\bar{k}j}\bar{\eta}^k = \theta^{*i}_{j}, \; \theta^{*i}_{\bar{k}\bar{h}}\bar{\eta}^k = 0,$$
$$\theta^{*i}_{kjr}\eta^k = -\theta^{*i}_{jr}, \; \theta^{*i}_{k\bar{h}j}\eta^k = 0, \; \theta^{*i}_{r\bar{k}j}\bar{\eta}^k = \theta^{*i}_{rj}, \; \theta^{*i}_{j\bar{k}\bar{h}}\bar{\eta}^k = 0,$$
$$\theta^{*i}_{k\bar{h}\bar{r}}\eta^k = \theta^{*i}_{\bar{h}\bar{r}}, \; \theta^{*j}_{hlrm}\eta^m = -2\theta^{*j}_{hlr}, \; \theta^{*j}_{h\bar{l}rm}\bar{\eta}^l = \theta^{*j}_{hrm},$$
$$\theta^{*j}_{hl\bar{r}m}\bar{\eta}^l = 0, \; \theta^{*j}_{h\bar{l}rm}\eta^m = -\theta^{*j}_{h\bar{l}r}, \; \theta^{*j}_{h\bar{l}\bar{r}m}\eta^m = 0, \; \theta^{*k}\eta_k = 0.$$

- Let $\tilde{F}$ be a complex Finsler metric on M; $\tilde{F} \rightsquigarrow \tilde{G}^i$ and $\tilde{\theta}^{*i}$.
- F and $\tilde{F}$ are projectively related iff $\exists$ P smooth on $T'M$, such that

$$\tilde{G}^i = G^i + V\eta^i \text{ and } \tilde{\theta}^{*i} = \theta^{*i} + Q\eta^i, \; i = \overline{1,n}, \tag{20}$$

  $V = (\dot{\partial}_k P)\eta^k$ is $(1,0)$-homogeneous, $Q = -2(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k$ is $(0,1)$-homogeneous and $P = V - \frac{1}{2}Q$.

4.2.2 Projective curvature invariants of Douglas type

Sketch of proof

- Differentiating in (20) w.r.t. η^j, we get

$$\tilde{G}^i = G^i + \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i}) + \frac{1}{n+1}(\overset{c}{\tilde{N}}{}^l_l - \overset{c}{N}{}^l_l)\eta^i - \frac{1}{2n}(\tilde{\theta}^{*l}_l - \theta^{*l}_l)\eta^i, \;\; i = \overline{1,n}. \tag{21}$$

- Successive differentiations of (21) w.r.t. η and $\bar{\eta}$ yield

$$\begin{aligned}
\tilde{G}^i_{jkh} &= G^i_{jkh} + \frac{1}{n+1}[(\dot{\partial}_h \tilde{D}_{jk} - \dot{\partial}_h D_{jk})\eta^i + \sum_{(k,j,h)} (\tilde{D}_{jh} - D_{jh})\delta^i_k] \\
&\quad + \frac{1}{2}(\tilde{\theta}^{*i}_{jkh} - \theta^{*i}_{jkh}) - \frac{1}{2n}[(\dot{\partial}_h \tilde{\theta}^{*l}_{ljk} - \dot{\partial}_h \theta^{*l}_{ljk})\eta^i + \sum_{(j,k,h)} (\tilde{\theta}^{*l}_{ljh} - \theta^{*l}_{ljh})\delta^i_k], \\
\tilde{G}^i_{j\bar{k}\bar{h}} &= G^i_{j\bar{k}\bar{h}} + \frac{1}{n+1}[(\dot{\partial}_j \tilde{D}_{\bar{k}\bar{h}} - \dot{\partial}_j D_{\bar{k}\bar{h}})\eta^i + (\tilde{D}_{\bar{k}\bar{h}} - D_{\bar{k}\bar{h}})\delta^i_j] \\
&\quad + \frac{1}{2}(\tilde{\theta}^{*i}_{j\bar{k}\bar{h}} - \theta^{*i}_{j\bar{k}\bar{h}}) - \frac{1}{2n}[(\dot{\partial}_{\bar{h}} \tilde{\theta}^{*l}_{l\bar{k}j} - \dot{\partial}_{\bar{h}} \theta^{*l}_{l\bar{k}j})\eta^i + (\tilde{\theta}^{*l}_{l\bar{k}\bar{h}} - \theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j], \\
\tilde{G}^i_{j\bar{k}h} &= G^i_{j\bar{k}h} + \frac{1}{n+1}[(\dot{\partial}_h \tilde{D}_{\bar{k}j} - \dot{\partial}_h D_{\bar{k}j})\eta^i + (\tilde{D}_{\bar{k}j} - D_{\bar{k}j})\delta^i_h + (\tilde{D}_{\bar{k}h} - D_{\bar{k}h})\delta^i_j] \\
&\quad + \frac{1}{2}(\tilde{\theta}^{*i}_{j\bar{k}h} - \theta^{*i}_{j\bar{k}h}) - \frac{1}{2n}[(\dot{\partial}_h \tilde{\theta}^{*l}_{l\bar{k}j} - \dot{\partial}_h \theta^{*l}_{l\bar{k}j})\eta^i + (\tilde{\theta}^{*l}_{l\bar{k}j} - \theta^{*l}_{l\bar{k}j})\delta^i_h + (\tilde{\theta}^{*l}_{l\bar{k}h} - \theta^{*l}_{l\bar{k}h})\delta^i_j],
\end{aligned} \tag{22}$$

where $D_{kh} = G^i_{ikh}$, $D_{\bar{k}\bar{h}} = G^i_{i\bar{k}\bar{h}}$ and $D_{\bar{k}h} = G^i_{i\bar{k}h}$ are respectively, hv-, $\bar{h}\bar{v}$- and $h\bar{v}$-*Ricci tensors.*

4.2.2 Projective curvature invariants of Douglas type $\rightsquigarrow$ complex Douglas space

$\Rightarrow$ 3 projective curvature invariants of Douglas type

$$D^i_{jkh} = G^i_{jkh} - \tfrac{1}{n+1}[(\dot{\partial}_h D_{jk})\eta^i + \textstyle\sum_{(k,j,h)} D_{jh}\delta^i_k] - \tfrac{1}{2}\{\theta^{*i}_{jkh} - \tfrac{1}{n}[(\dot{\partial}_h \theta^{*l}_{ljk})\eta^i + \textstyle\sum_{(j,k,h)} \theta^{*l}_{ljh}\delta^i_k]\},$$

$$D^i_{j\bar{k}\bar{h}} = G^i_{j\bar{k}\bar{h}} - \tfrac{1}{n+1}[(\dot{\partial}_j D_{\bar{k}\bar{h}})\eta^i + D_{\bar{k}\bar{h}}\delta^i_j] - \tfrac{1}{2}\{\theta^{*i}_{j\bar{k}\bar{h}} - \tfrac{1}{n}[(\dot{\partial}_{\bar{h}} \theta^{*l}_{l\bar{k}j})\eta^i + \theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j]\},$$

$$D^i_{j\bar{k}h} = G^i_{j\bar{k}h} - \tfrac{1}{n+1}[(\dot{\partial}_h D_{\bar{k}j})\eta^i + D_{\bar{k}j}\delta^i_h + D_{\bar{k}h}\delta^i_j] - \tfrac{1}{2}\{\theta^{*i}_{j\bar{k}h} - \tfrac{1}{n}[(\dot{\partial}_h \theta^{*l}_{l\bar{k}j})\eta^i + \theta^{*l}_{l\bar{k}j}\delta^i_h + \theta^{*l}_{l\bar{k}h}\delta^i_j]\}. \tag{23}$$

Definition

A complex Finsler space (M, F) is called a complex Douglas space if all of the invariants (23) are vanishing.

Remark

If F is Kähler-Berwald, then the projective curvature invariants of Douglas type are vanishing $\Rightarrow$ any Kähler-Berwald space is a complex Douglas space.

4.2.2 Projective curvature invariants of Douglas type $\rightsquigarrow$ complex Douglas space

Theorem

Let F and $\tilde{F}$ be projectively related complex Finsler metrics on M. F is a Douglas metric if and only if $\tilde{F}$ is a Douglas metric.

Theorem

Let (M, F) be a complex Finsler space. (M, F) is a complex Douglas space if and only if it is generalized Berwald with

$$\theta^{*i}_{jkh} = \frac{1}{n}[(\dot{\partial}_h \theta^{*l}_{ljk})\eta^i + \sum_{(j,k,h)} \theta^{*l}_{ljh}\delta^i_k], \tag{24}$$

$$\theta^{*i}_{j\bar{k}\bar{h}} = \frac{1}{n}[(\dot{\partial}_{\bar{h}} \theta^{*l}_{lj\bar{k}})\eta^i + \theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j],$$

$$\theta^{*i}_{j\bar{k}h} = \frac{1}{n}[(\dot{\partial}_h \theta^{*l}_{lj\bar{k}})\eta^i + \theta^{*l}_{lj\bar{k}}\delta^i_h + \theta^{*l}_{l\bar{k}h}\delta^i_j].$$

- We call generalized Kähler the complex Finsler spaces which satisfy (24).
- Notation $K^i = \theta^{*i} - \frac{1}{n}\theta^{*l}_l\eta^i$.

4.2.2 Projective curvature invariants of Douglas type $\rightsquigarrow$ complex Douglas space

Theorem

Let (M, F) be a complex Finsler space. (M, F) is a generalized Kähler space if and only if the functions K^i are homogeneous polynomials in η and in $\bar{\eta}$ of first degree. Moreover, the functions K^i vanish identically if and only if the space is weakly Kähler.

Proposition

Let (M, F) be a complex Finsler space. If θ^{*i} are homogeneous polynomials in η and $\bar{\eta}$ of first degree, then (M, F) is a generalized Kähler space.

- Examples of generalized Kähler metrics are provided by the pure Hermitian metrics.

$$g_{i\bar{j}} = g_{i\bar{j}}(z) \quad \Rightarrow \quad G^i = \frac{1}{2} g^{\bar{m}i} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l \eta^j \text{ and } \theta^{*i} = -g^{\bar{m}i} \left(\frac{\partial g_{l\bar{m}}}{\partial \bar{z}^k} - \frac{\partial g_{l\bar{k}}}{\partial \bar{z}^m}\right) \eta^l \bar{\eta}^k.$$

- The pure Hermitian metrics are complex Douglas metrics.

Corollary

Let (M, F) be a complex Finsler space. (M, F) is a complex Douglas space if and only if it is a generalized Berwald space and $K^i = \varphi^i_{\bar{r}s} \bar{\eta}^r \eta^s$, where $\varphi^i_{\bar{r}s}$ are smooth functions that depend only on z and $\bar{z}$.

4.2.3 Weakly Kähler projective changes $\rightsquigarrow$ proj. curvature invariant of Weyl type

Let F be a weakly Kähler Finsler metric. Then, the projective changes is

$$\tilde{G}^i = G^i + P\eta^i, \tag{25}$$

where P is a $(1,0)$-homogeneous and the weakly Kähler property is preserved. Assuming that F is generalized Berwald $\Rightarrow$

1. F and $\tilde{F}$ are Kähler-Berwald metrics;
2. $K^i_{\bar{j}\bar{k}h} = 0$, $K^i_{j\bar{k}h} = -\delta_{\bar{k}} L^i_{jh}$ and $R^i_{j\bar{k}h} = K^i_{j\bar{k}h} + K^l_{m\bar{k}h}\eta^m C^i_{jl}$;
3. $R_{\bar{r}j\bar{k}h} = K_{\bar{r}j\bar{k}h} + K^l_{m\bar{k}h}\eta^m C_{j\bar{r}l}$ and $\mathcal{K}_F(z,\eta) = \frac{2}{L^2} K_{\bar{r}j\bar{k}h}\bar{\eta}^r\eta^j\bar{\eta}^k\eta^h$;
4. P is holomorphic with respect to η, i.e. $P_{\bar{k}} = 0$.
   Consequently,

$$\tilde{K}^i_{j\bar{k}h} = K^i_{j\bar{k}h} - P_{jh|\bar{k}}\eta^i - P_{j|\bar{k}}\delta^i_h - P_{h|\bar{k}}\delta^i_j, \tag{26}$$
$$0 = P_{jhr|\bar{k}}\eta^i + P_{jh|\bar{k}}\delta^i_r + P_{jr|\bar{k}}\delta^i_h + P_{hr|\bar{k}}\delta^i_j.$$

$\Rightarrow$ the projective curvature invariant of the **Weyl type**,

$$W^i_{j\bar{k}h} = K^i_{j\bar{k}h} - \frac{1}{n+1}(K_{\bar{k}j}\delta^i_h + K_{\bar{k}h}\delta^i_j), \tag{27}$$

where $K_{\bar{k}h} = K^i_{i\bar{k}h}$ is $h\bar{h}$-Ricci tensor.

4.3 Locally projectively flat complex Finsler metrics

Theorem

Let (M,F) be a connected n-dimensional Kähler-Berwald spaces, $n \geq 2$. Then, $W^i_{j\bar{k}h} = 0$ if and only if $K_{\bar{m}j\bar{k}h} = \frac{\mathcal{K}_F}{4}(g_{j\bar{k}}g_{h\bar{m}} + g_{h\bar{k}}g_{j\bar{m}})$. In this case, $\mathcal{K}_F = c$, where c is a constant on M and the space is either pure Hermitian with $K_{\bar{k}j} = \frac{c(n+1)}{4}g_{j\bar{k}}$ or non-pure Hermitian with $c = 0$ and $K^i_{j\bar{k}h} = 0$.

- Let $\tilde{F}$ be a locally Minkowski complex Finsler metric on M, (i.e. $\forall\ z \in M$, $\exists$ local charts such that the fundamental metric tensor $\tilde{g}_{i\bar{j}}$ depends only on η) $\Rightarrow \tilde{G}^i = \tilde{\theta}^{*i} = 0$.
- A complex Finsler metrics F on M is called locally projectively flat iff it is projectively related to the locally Minkowski metric $\tilde{F}$.

Theorem

Let (M,F) be a connected n-dimensional complex Finsler space, $n \geq 2$. If F is locally projectively flat then it has constant holomorphic curvature. Moreover, if the constant value of the holomorphic curvature is non-zero, then (M,F) is a pure Hermitian space.

Theorem [N.A, P. Kopacz, Diff. Geom. Appl. 2017]

Let F be a complex Finsler metric on domain D from $\mathbb{C}^n$. F is locally projectively flat if and only if it is Kähler-Berwald and $G^i = \frac{1}{F}\frac{\partial F}{\partial z^k}\eta^k\eta^i$.

4.5 Complex Douglas spaces with Randers metrics

- Consider $\tilde{a} = a_{i\bar{j}}(z)dz^i \otimes d\bar{z}^j$ a pure Hermitian metric and $b = b_i(z)dz^i$ a differential $(1,0)$-form $\rightsquigarrow$ the complex Randers metric

$$F(z,\eta) = \alpha + |\beta|, \tag{28}$$

$\alpha(z,\eta) = \sqrt{a_{i\bar{j}}\eta^i\bar{\eta}^j}$ and $\beta(z,\eta) = b_i\eta^i$ [N.A., G. Munteanu, J. Kor. Math. Soc. 2009];

- $G^i = \overset{a}{G^i} + \frac{1}{2\gamma}(l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j} - \frac{\beta^2}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\xi^i\eta^j + \frac{\beta}{4|\beta|}k^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j$;
- $\overset{a}{\theta^{*i}} = -\Gamma_{l\bar{r}\bar{m}}a^{\bar{m}i}\eta^l\bar{\eta}^r$, $\Gamma_{l\bar{r}\bar{m}} = \frac{\partial a_{l\bar{m}}}{\partial \bar{z}^r} - \frac{\partial a_{l\bar{r}}}{\partial \bar{z}^m}$;
- $\theta^{*i} = -\alpha(\Gamma_{l\bar{r}\bar{m}}\tau^l\bar{\eta}^r + \frac{2\beta}{|\beta|}\Omega_{\bar{m}})(h^{\bar{m}i} - \frac{\bar{\beta}}{\gamma}b^{\bar{m}}\eta^i)$,

  where $h^{\bar{m}i} = a^{\bar{m}i} - \frac{\alpha^2}{\gamma}b^{\bar{m}}b^i$ and $\Omega_{\bar{m}} = \overset{a}{N^{\bar{s}}_{\bar{m}}}b_{\bar{s}} - \frac{\partial b_{\bar{r}}}{\partial \bar{z}^m}\bar{\eta}^r - \frac{\bar{\beta}^2}{|\beta|^2}\frac{\partial b_l}{\partial \bar{z}^m}\eta^l$.

Theorem [N.A., G. Munteanu, J. Geom. Phys. 2012]

Let (M,F) be a connected complex Randers space. Then, (M,F) is a generalized Berwald space if and only if $(\bar{\beta}l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j} + \beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j = 0$. Moreover, given any of them, $G^i = \overset{a}{G^i}$.

4.5 Complex Douglas spaces with Randers metrics

Theorem

Let (M,F) be a connected complex Randers space. Then, (M,F) is a complex Douglas space if and only if $(\bar{\beta}l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=0$ and $K^i=\overset{a}{K^i}$. Given any of them, $\Omega_{\bar{m}}=-\frac{1}{2}\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$. Moreover, if α is Kähler, then (M,F) is a Kähler-Berwald space.

Theorem

Let (M,F) be a connected complex Randers space. If (M,F) is a generalized Berwald space, then $K^i=\overset{a}{K^i}$ if and only if $\theta^{*i}=\overset{a}{\theta^{*i}}$.

Theorem

Let (M,F) be a connected complex Randers space. Then, (M,F) is a complex Douglas space if and only if $(\bar{\beta}l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=0$ and $\theta^{*i}=\overset{a}{\theta^{*i}}$.

Theorem [N.A., G. Munteanu, Nonlinear Anal. Real World Appl. 2012]

Let (M,F) be a connected complex Randers space. Then, α and F are projectively related if and only if F is generalized Berwald and $B^i=-P\eta^i$, for any $i=\overline{1,n}$, where $P=-\frac{\bar{\beta}}{4F|\beta|}\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}}\eta^l\bar{\eta}^r$.

4.5 Complex Douglas spaces with Randers metrics

Theorem

Let (M, F) be a connected complex Randers space. Then, (M, F) is a complex Douglas space if and only if α and F are projectively related.

Remark

Any complex Randers Douglas space of dimension two is a Kähler-Berwald space.

Example 1. Let $\Delta = \left\{(z,w) \in \mathbf{C}^2,\ |w| < |z| < 1\right\}$ be the Hartogs triangle with the pure Kähler-Hermitian metric

$$a_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\log \frac{1}{(1-|z|^2)(|z|^2-|w|^2)}),\ (z,w) = (z^1, z^2). \quad (29)$$

and $b_1 = \frac{w}{|z|^2-|w|^2},\ b_2 = -\frac{z}{|z|^2-|w|^2}$ $\Rightarrow$ a Kähler-Berwald metric $F = \alpha + |\beta| \rightsquigarrow \alpha$ and F are projectively related.

Example 2. On $M = \mathbf{C}^3$ we set: the pure Hermitian metric

$$\alpha^2 = e^{z^1+\bar{z}^1}\left|\eta^1\right|^2 + e^{z^2+\bar{z}^2}\left|\eta^2\right|^2 + e^{z^1+\bar{z}^1+z^3+\bar{z}^3}\left|\eta^3\right|^2 \quad (30)$$

and the $(1,0)$-differential form β given by $\beta = e^{z^2}\eta^2$ $\Rightarrow$ a complex Douglas-Randers metric $F = \alpha + |\beta| \rightsquigarrow \alpha$ and F are projectively related.

4.5 Complex Douglas spaces with Randers metrics

Example 3. On Hartogs triangle Δ, we set $b_1 = \frac{w}{|z|^2-|w|^2}$ and $b_2 = -\frac{z}{|z|^2-|w|^2}$ and the pure Hermitian metric $\alpha^2 = a_{i\bar{j}}\eta^i\overline{\eta}^j$, with $(a_{i\bar{j}}) = \begin{pmatrix} \frac{1}{1-|z|^2} + b_1 b_{\bar{1}} & b_1 b_{\bar{2}} \\ b_2 b_{\bar{1}} & b_2 b_{\bar{2}} \end{pmatrix}$

$\Rightarrow$ a generalized Berwald-Randers metric $F = \alpha + |\beta|$ which is not Douglas.

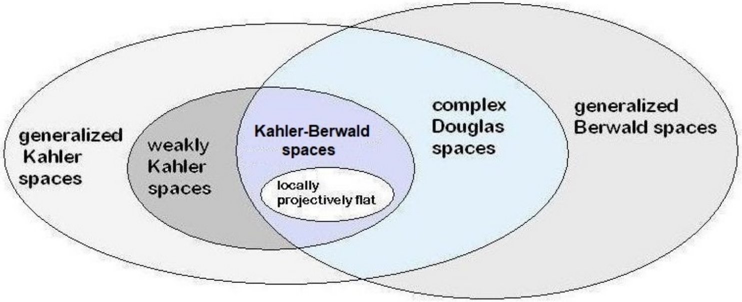


Figure 2: Inclusion diagram

5.2 Rudiments of $\mathbb{R}$-complex Finsler geometry

N. Aldea, Zermelo deformation of Hermitian metrics by holomorphic vector fields. *Results in Math.*, 75(4) (2020), 140.

N. Aldea, P. Kopacz, Generalized Zermelo navigation on Hermitian manifolds under mild wind. *Diff. Geom. Appl.*, 54(A) (2017), 325-343.

Definition

An $\mathbb{R}$-complex Finsler space is a pair (M, F), where F is a continuous function $F : T'M \longrightarrow \mathbb{R}_+$ satisfying the conditions:
i) $L = F^2$ is smooth on $\widetilde{M} = T'M \backslash \{0\}$;
ii) $F(z, \eta) \geq 0$ for all $(z, \eta) \in T'M$; the equality holds if and only if $\eta = 0$;
iii) $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = \lambda F(z, \eta, \bar{z}, \bar{\eta})$, for all $\lambda > 0$.

- The Hessian and the Levi matrices of L induce the tensors

$$g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}, \quad g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}, \quad g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \bar{\eta}^j} \rightsquigarrow \tag{31}$$

$$(\dot{\partial}_i L)\eta^i + (\dot{\partial}_{\bar{i}} L)\bar{\eta}^i = 2L, \qquad g_{ij}\eta^i + g_{j\bar{i}}\bar{\eta}^i = \dot{\partial}_j L, \qquad L = \mathrm{Re}\{g_{ij}\eta^i\eta^j\} + g_{i\bar{j}}\eta^i\bar{\eta}^j,$$

$$(\dot{\partial}_j g_{ik})\eta^j + (\dot{\partial}_{\bar{j}} g_{ik})\bar{\eta}^j = 0, \qquad (\dot{\partial}_j g_{i\bar{k}})\eta^j + (\dot{\partial}_{\bar{j}} g_{i\bar{k}})\bar{\eta}^j = 0.$$

$\Rightarrow$ two general classes of $\mathbb{R}$-complex Finsler spaces:

1. *$\mathbb{R}$-complex Hermitian Finsler spaces*, i.e. $(g_{i\bar{j}})$ is positive definite
2. *$\mathbb{R}$-complex non-Hermitian Finsler spaces*, i.e. (g_{ij}) is positive definite

5.2 Rudiments of $\mathbb{R}$-complex Finsler geometry

Let (M, F) be a $\mathbb{R}$-complex Hermitian Finsler space.

- Chern-Finsler (c.n.c) with $N_j^i = g^{\bar{m}i} \frac{\partial^2 L}{\partial z^k \partial \eta^m} = g^{\bar{m}i}[(\partial_k g_{\bar{r}\bar{m}})\bar{\eta}^r + (\partial_k g_{s\bar{m}})\eta^s]$.
- Chern-Finsler connection $D : \Gamma(T'\widetilde{M}) \to \Gamma(T^*_{\mathbb{C}}\widetilde{M} \otimes T'\widetilde{M})$ locally given by

$$L^i_{jk} = g^{\bar{m}i}(\delta_j g_{k\bar{m}}), \quad C^i_{jk} = g^{\bar{m}i}(\dot{\partial}_j g_{k\bar{m}}), \quad L^i_{j\bar{k}} = C^i_{j\bar{k}} = 0, \tag{32}$$

  with $L^i_{jk} = \dot{\partial}_j N^i_k$ and $N^i_k = L^i_{jk}\eta^j + (\dot{\partial}_{\bar{r}} N^i_k)\bar{\eta}^r$ and $T^i_{jk} = L^i_{jk} - L^i_{kj}$.
- $\mathcal{K}_F(z,\eta) = \frac{2}{\tilde{L}^2} g_{i\bar{m}} R^i_{\bar{j}\bar{h}k} \eta^j \bar{\eta}^h \eta^k \bar{\eta}^m$ is the holomorphic curvature of F in direction η, where $R^i_{\bar{j}\bar{h}k} = -\delta_{\bar{h}} L^i_{jk} - (\delta_{\bar{h}} N^l_k) C^i_{jl}$ and $\tilde{L} = g_{i\bar{m}}\eta^k\bar{\eta}^m$.
- (M, F) is called **strongly Kähler** or **Kähler** iff $T^i_{jk} = 0$ or $T^i_{jk}\eta^j = 0$, respectively.
- $\alpha^2 = \mathrm{Re}\{a_{ij}\eta^i\eta^j\} + a_{i\bar{j}}\eta^i\bar{\eta}^j$, with $a = a_{i\bar{j}}(z)\, dz^i \otimes d\bar{z}^j$ a Hermitian metric on M.
- $\beta = \mathrm{Re}\{b_i\eta^i\}$, with $b = b_i(z)dz^i$ a differential $(1,0)$-form, $||b||^2 = a^{\bar{j}i} b_i b_{\bar{j}}$.

Lemma

Let $F = \alpha + \beta$ be an $\mathbb{R}$-complex Randers function with $a_{ij} = \frac{1}{2} b_i b_j$. Then, F is positive on $\widetilde{M}$ if and only if $||b||^2 < 2$. Moreover, any of these assertions implies $\alpha^2 - \beta^2 > 0$.

- $||b||^2 < 2$ also assures that $g_{i\bar{j}}$ is positive definite $\rightsquigarrow$ $F = \alpha + \beta$ with $a_{ij} = \frac{1}{2} b_i b_j$ is an $\mathbb{R}$-complex Hermitian Randers metric (briefly $\mathbb{R}$-complex Randers metric).

5.3 Generalized Zermelo navigation under weak wind

- (M, h) is an n-dimensional Hermitian manifold ($h = h_{i\bar{j}} dz^i \otimes d\bar{z}^j$ is a Hermitian metric, $h_{j\bar{k}}(z) = h(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k})) \rightsquigarrow$ the imaginary sea;
- $\rightsquigarrow$ a perturbing vector field $W = W^j \frac{\partial}{\partial z^j}$, $||W||_h < 1 \rightsquigarrow$ a weak wind;
- u is the velocity of a ship in the absence of wind, $||u||_h = f(z) \in (||W||_h, 1]$, where $f : M \to (||W||_h, 1]$ is smooth $\rightsquigarrow$ the ship's speed $||u||_h$ is space-dependent;
- $(h, f(z), W) \rightsquigarrow$ **the generalized navigation data**

$\Rightarrow$ the ship's resultant velocity $v = u + W \Rightarrow h$ is deformed into the $\mathbb{R}$-complex Randers metric $F(z, \eta) = \alpha + \beta$ (called **W-Zermelo deformation**) with

$$\alpha = \sqrt{\frac{[\mathrm{Re}\, h(\eta, \bar{W})]^2 + ||\eta||_h^2 \psi}{\psi^2}} = \sqrt{\mathrm{Re}\{a_{ij}\eta^i\eta^j\} + a_{i\bar{j}}\eta^i\bar{\eta}^j}, \quad \beta = -\frac{\mathrm{Re}\, h(\eta, \bar{W})}{\psi} = \mathrm{Re}\{b_i\eta^i\},$$

α is a Hermitian metric, $\mathrm{Re}\, h(\eta, \bar{W}) \neq 0$, $||b||^2 \in (0, 2)$ because the wind W is weak,

$$a_{i\bar{j}} = \frac{h_{i\bar{j}}}{\psi} + \frac{W_i W_{\bar{j}}}{2\psi^2}, \quad a_{ij} = \frac{W_i W_j}{2\psi^2}, \quad b_i = -\frac{W_i}{\psi}, \quad \psi = f^2 - ||W||_h^2. \tag{33}$$

Theorem

An $\mathbb{R}$-complex Hermitian Finsler metric F is of Randers type, i.e. $F = \alpha + \beta$ with (33), if and only if it solves the generalized Zermelo navigation problem on a Hermitian manifold (M, h), with space-dependent ship's speed $||u(z)||_h \leq 1$ and under action of weak wind W. Moreover, F is a pure Hermitian metric conformal to h, with the conformal factor $\frac{1}{||u(z)||_h}$, if and only if $W = 0$.

5.4 W-Zermelo metric $\rightsquigarrow$ 5.4.2 Holomorphic Zermelo deformation

- W is called f-holomorphic if $\frac{\partial W^k}{\partial \bar{z}^r} = 0$ and $W^k_{|j} = \frac{\partial \log f^2}{\partial z^j} W^k$, where $W^k_{|j} = \frac{\partial W^k}{\partial z^j} + h^{\bar{m}k} \frac{\partial h_{r\bar{m}}}{\partial z^j} W^r$, $k = \overline{1, n}$.
- W is f-holomorphic iff $\overset{a}{\delta}_j \beta = 0$, with $b_i = -\frac{W_i}{\psi}$.

Theorem

Let (M, h) be an n-dimensional Kähler manifold, $n \geq 2$, and $(h, f(z), W)$ be the generalized navigation data, with W an f-holomorphic vector field. Then, f is a constant if and only if W-Zermelo deformation F is strongly Kähler.

Theorem

Let (M, h) be an n-dimensional Hermitian manifold and $(h, f(z), W)$ be the generalized navigation data, with W an f-holomorphic vector field. Then, the holomorphic curvature in direction η, corresponding to W-Zermelo deformation F is

$$\mathcal{K}_F(z, \eta) = \frac{\tilde{h}^4 P}{(1-c) f^2} \left(\mathcal{K}_h(z, \eta) + \frac{2}{\tilde{h}^2} \frac{\partial^2 \log f^2}{\partial z^j \partial \bar{z}^m} \eta^j \bar{\eta}^m \right), \tag{34}$$

where $c \in (0, 1)$. If $\frac{\partial \log f^2}{\partial z^j}$ is a holomorphic function, then $\mathcal{K}_F(z, \eta) = \frac{\tilde{h}^4 P}{(1-c) f^2} \mathcal{K}_h(z, \eta)$.

Genesis of Matsumoto's slope-of-a-mountain problem

Finsler's answer to Matsumoto's letter about "models of Finsler spaces":

(...) on a slope of the earth surface we sometimes measure the distance in a time (...) The shortest line along which we can reach the goal, for instance, the top of a mountain as soon as possible will be a complicated curve. (P. Finsler, 1969)

M. Matsumoto, A slope of a mountain is a Finsler surface with respect to a time measure. *J. Math. Kyoto Univ.*, 29 (1989), 17-25.

Matsumoto's slope-of-a-mountain problem:

Suppose a person walking on a horizontal plane with velocity c, while the gravitational force is acting perpendicularly on this plane. The person is almost ignorant of the action of this force. Imagine the person walks now with same velocity on the inclined plane of angle ε to the horizontal sea level.

Under the influence of gravitational forces, what is the trajectory the person should walk in the centre to reach a given destination in the shortest time?

$\rightsquigarrow$ an exact formulation of the model of a Finsler surface

$\rightsquigarrow$ the most efficient (time-minimizing) paths are the geodesics of the slope metric.

General view

Aim: to present a general time-optimal navigation problem on a slippery mountain slope under the action of gravity which unifies all extensions of Matsumoto's slope-of-a-mountain problem (MAT) and, in particular, links MAT and Zermelo's navigation problem (ZNP).

Approached problems:

- One-parameter time-optimal navigation problems through the slippery slope models that incorporate a traction coefficient:

  ▶ slippery with a cross-traction coefficient $\eta \in [0, 1]$ (SLIPPERY), interlinking MAT and ZNP under the influence of the gravitational wind (Ch. 7);

  ▶ slippery with an along-traction coefficient $\tilde{\eta} \in [0, 1]$ (S-CROSS) which connects the cross problem under cross-gravity effect (CROSS) and ZNP under the influence of the gravitational wind (Ch. 8);

- A two-parameter time-optimal navigation problem through a slippery slope model where both traction coefficients η and $\tilde{\eta}$ are admitted to vary simultaneously, in full ranges (Ch. 9).

6.1 Finsler manifolds

Main refs.: S.S. Chern & Z. Shen 2005, I. Bucătaru & R. Miron 2007, Z. Shen CJM 2003, C. Yu & H. Zhu DGA 2011.

- M - a real n-dimensional C^∞-manifold, $n > 1$;
- (x^i), $i = 1, ..., n$ - the local coordinates in $x \in M$;
- T_xM - the tangent space at $x \in M$;
- $\left\{\frac{\partial}{\partial x^i}\right\}$ - the natural basis for the tangent bundle $TM = \bigcup\limits_{x\in M} T_xM$;
- $\forall y \in T_xM$, $y = y^i \frac{\partial}{\partial x^i}$; (x^i, y^i) - the local coordinates in $(x, y) \in TM$.

Definition

The pair (M, F) is a real Finsler manifold if $F : TM \to [0, \infty)$ is a continuous function with the following properties:
$i)$ F is a C^∞-function on the slit tangent bundle $TM_0 = TM\backslash\{0\}$;
$ii)$ F is positively homogeneous of degree one with respect to y, i.e. $F(x, cy) = cF(x, y)$, for all $c > 0$;
$iii)$ the Hessian $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite for all $(x, y) \in TM_0$.

6.1 Finsler manifolds

- $I_F = \{(x,y) \in TM \mid F(x,y) = 1\}$ denotes the indicatrix of F;
- $iii)$ refers to the fact that I_F is strongly convex;
- $S = y^i \frac{\partial}{\partial x^i} - 2\mathcal{G}^i \frac{\partial}{\partial y^i}$, is a spray on M;
- The spray coefficients $\mathcal{G}^i = \mathcal{G}^i(x,y)$, $i = 1,...,n$ are positively homogeneous of degree two with respect to y;
- If S is induced by a Finsler metric F, then

$$\mathcal{G}^i(x,y) = \frac{1}{4} g^{il} \{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\} = \frac{1}{4} g^{il} \left(2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) y^j y^k, \quad (35)$$

  (g^{il}) being the inverse matrix of (g_{il});
- $\gamma : [0,1] \to M$, $\gamma(t) = (\gamma^i(t))$, is a regular piecewise C^∞-curve on M;
- γ is F-geodesic if $\dot{\gamma}(t) = \frac{d\gamma}{dt}$ is parallel along the curve, i.e. in the local coordinates, $\gamma^i(t)$, $i = 1,...,n$ are the solutions of the ODE system

$$\ddot{\gamma}^i(t) + 2\mathcal{G}^i(\gamma(t), \dot{\gamma}(t)) = 0. \quad (36)$$

6.1 Finsler manifolds

Proposition [Z. Shen, Canad. J. Math. 2003]

Let (M, F) be a Finsler manifold and W a vector field on M such that $F(x, -W) < 1$. Then the solution of the Zermelo's navigation problem with the navigation data (F, W) is a Finsler metric $\tilde{F}$ obtained by solving the equation

$$F(x, y - \tilde{F}(x, y)W) = \tilde{F}(x, y), \tag{37}$$

for any nonzero $y \in T_xM$, $x \in M$.

Lemma [S.S. Chern, Z. Shen, 2005]

Let (M, F) be a Finsler manifold and W be a vector field on M with $F(x, -W) < 1$ $\forall x \in M$. Define $\tilde{F} : TM \to [0, \infty)$ by (37). For any piecewise C^∞-curve γ on M, the $\tilde{F}$-length of γ is equal to the time for which the object travels along it.

Remark

Any regular piecewise C^∞-curve $\gamma : [0, 1] \to M$, parametrized by time, that represents a trajectory in Zermelo's navigation problem has unit $\tilde{F}$-length, i.e. $\tilde{F}(\gamma(t), \dot{\gamma}(t)) = 1$.

6.2 General (α, β)-metrics

- A Finsler metric F is called general (α, β)-metric if it can be expressed as

$$F = \alpha\phi(b^2, s),$$

where $\phi(b^2, s)$ is a positive C^∞-function in the variables

$$b^2 = ||\beta||_\alpha^2 = a^{ij} b_i b_j \quad s = \frac{\beta}{\alpha},$$

with $|s| \leq b < b_0$ and $0 < b_0 \leq \infty$; see [C. Yu, H. Zhu, Diff. Geom. Appl. 2011].

Proposition 1 [C. Yu, H. Zhu, Diff. Geom. Appl. 2011]

Let M be an n-dimensional manifold. $F = \alpha\phi(b^2, s)$ is a Finsler metric for any Riemannian metric α and 1-form β, with $||\beta||_\alpha < b_0$ if and only if $\phi = \phi(b^2, s)$ is a positive C^∞-function satisfying

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0,$$

when $n \geq 3$ or

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0,$$

when $n = 2$, where $s = \frac{\beta}{\alpha}$ and $b = ||\beta||_\alpha$ satisfy $|s| \leq b < b_0$.

6.2 General (α, β)-metrics

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad r_i = b^j r_{ji}, \quad r^i = a^{ij} r_j, \quad r_{00} = r_{ij} y^i y^j, \quad r_0 = r_i y^i, \quad r = b^i r_i,$$

$$s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_i = b^j s_{ji}, \quad s^i = a^{ij} s_j, \quad s^i_0 = a^{ij} s_{jk} y^k, \quad s_0 = s_i y^i,$$

with $b^j = a^{ji} b_i$, $b_{i|j} = \frac{\partial b_i}{\partial x^j} - \Gamma^k_{ij} b_k$ and $\Gamma^k_{ij} = \frac{1}{2} a^{km} \left(\frac{\partial a_{jm}}{\partial x^i} + \frac{\partial a_{im}}{\partial x^j} - \frac{\partial a_{ij}}{\partial x^m} \right)$.

Proposition 2 [C. Yu, H. Shu, Diff. Geom. Appl. 2011]

For a general (α, β)-metric $F = \alpha\phi(b^2, s)$, its spray coefficients $\mathcal{G}^i$ are related to the spray coefficients $\mathcal{G}^i_\alpha$ of α by

$$\begin{aligned} \mathcal{G}^i &= \mathcal{G}^i_\alpha + \alpha Q s^i_0 + \left[\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Omega(r_0 + s_0)\right] \frac{y^i}{\alpha} \\ &\quad + \left[\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Pi(r_0 + s_0)\right] b^i - \alpha^2 R(r^i + s^i), \end{aligned}$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2}, \qquad \Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]},$$

$$\Psi = \frac{\phi_{22}}{2[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]}, \qquad \Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]},$$

$$\Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi, \qquad R = \frac{\phi_1}{\phi - s\phi_2}.$$

7.1 Slippery slope model $\rightsquigarrow$ 7.1.1 Gravitational wind

N. Aldea, P. Kopacz, Time geodesics on a slippery slope under gravitational wind. *Nonlinear Anal.-Theor.* 227 (2023), 113160.

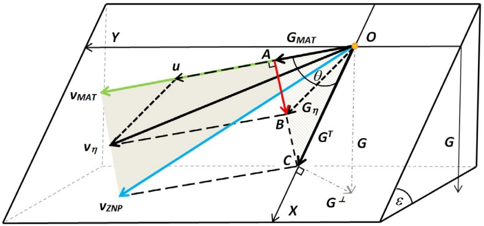
- (M, h) - a surface embedded in $\mathbb{R}^3 \rightsquigarrow$ a mountain slope;
- π_O - the tangent plane to M at an arbitrary point $O \in M$;
- $\mathbf{G}$ - a gravitational field in $\mathbb{R}^3$ that affects M;
- $\mathbf{G} = \mathbf{G}^T + \mathbf{G}^\perp \rightsquigarrow \mathbf{G}^\perp$ is orthogonal to M and $\mathbf{G}^T$ is tangent to M in O;
- $\mathbf{G}^T$ - **gravitational wind** $\rightsquigarrow$

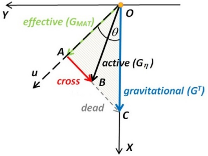
  ▶ $\mathbf{G}^T$ acts along an anti-gradient (a negative gradient), i.e. the steepest descent (downhill) direction;

  ▶ $\mathbf{G}^T$ depends on the gradient vector field related to the slope M and a given acceleration of gravity.

  ▶ $||\mathbf{G}^T||_h = \sqrt{h(\mathbf{G}^T, \mathbf{G}^T)}$ - **force** of $\mathbf{G}^T$;
- $u \in \pi_O$ - a desired direction of motion $\rightsquigarrow$ the self-velocity of a moving craft or a walker on M.

7.1 Slippery slope model ⇝ 7.1.1 Gravitational wind





$$\mathbf{G}^T = \underbrace{\text{Proj}_u \mathbf{G}^T}_{\substack{\mathbf{G}_{MAT} \\ \text{along-gravity effect}}} + \underbrace{\text{Proj}_{u^\perp} \mathbf{G}^T}_{\substack{\mathbf{G}^\perp{}_{MAT} \\ \text{cross-gravity effect}}}$$

Remark

Cross-gravity effect is omitted in Matsumoto's reasoning: *the component perpendicular to the velocity u* ***is regarded to be cancelled*** *by planting the walker's legs on the road determined by u* [M. Matsumoto, J. Math. Kyoto Univ. 1989].

In MAT the **resultant velocity:** $v_{MAT} = u + \mathbf{G}_{MAT} \quad \rightsquigarrow \quad v_{MAT} \parallel u.$

7.1 Slippery slope model $\rightsquigarrow$ 7.1.1 Gravitational wind

- $\rightsquigarrow$ **cross-traction coefficient** $\eta \in [0,1] \rightsquigarrow \overrightarrow{AB} = (1-\eta)\,\mathrm{Proj}_{u^\perp}\mathbf{G}^T$
- $\rightsquigarrow$ **active wind** $\mathbf{G}_\eta = \mathrm{Proj}_u\mathbf{G}^T + \overrightarrow{AB} = \mathrm{Proj}_u\mathbf{G}^T + (1-\eta)\mathrm{Proj}_{u^\perp}\mathbf{G}^T$

$$\Rightarrow \quad \mathbf{G}_\eta = \underbrace{\eta\,\mathbf{G}_{MAT}}_{\text{I. } u\text{-direction-dependent deformation}} + \underbrace{(1-\eta)\,\mathbf{G}^T}_{\text{II. rigid translation}} \tag{38}$$

- **Resultant velocity:** $v_\eta = u + \mathbf{G}_\eta \rightsquigarrow v_\eta \not\parallel u$ in contrast to MAT;
- $||\overrightarrow{AB}||_h$ - measures the sliding effect $\rightsquigarrow$ the lesser the traction, the greater the sliding; $\eta = 1$ in MAT and $\eta = 0$ in ZNP under action of $\mathbf{G}^T$.

SLIPPERY problem: *Suppose a person walks on a horizontal plane at a constant speed, while the gravity acts orthogonally to this plane. Imagine the person endeavours to walk now on the slippery mountainside with a given traction coefficient and under the influence of gravity.*

How should the person navigate on the slippery slope of a mountain in order to travel from one point to another in the shortest time?

7.1 Slippery slope model ⇝ 7.1.2 Main results ⇝ Background

- (M, h) - an n-dimensional Riemannian manifold, $n > 1 \rightsquigarrow$ model for a slippery slope of a mountain.
- $\omega^\sharp = h^{ji} \frac{\partial p}{\partial x^j} \frac{\partial}{\partial x^i}$ - the gradient vector field of p; $p : M \to \mathbb{R}$ is a C^∞-function on M.
- $\mathbf{G}^T = -\bar{g}\omega^\sharp$ is the gravitational wind ($\bar{g}$ is the rescaled magnitude of the acceleration of gravity g, i.e. $\bar{g} = \lambda g$, $\lambda > 0$).
- Based on scaling, for the self-velocity u of a moving craft on the slope we assume that $||u||_h = \sqrt{h(u, u)} = 1$.
- Notations:

$$\alpha^2 = ||y||_h^2 = h_{ij} y^i y^j \quad \text{and} \quad \beta = -\frac{1}{\bar{g}} h(y, \mathbf{G}^T) = h(y, \omega^\sharp) = b_i y^i, \tag{39}$$

$\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ and $||\beta||_h = ||\omega^\sharp||_h$, $(x, y) \in TM$.

- $\Rightarrow \beta$ - closed differential 1-form, i.e. $s_{ij} = 0$ [N.A, P. Kopacz, R. Wolak, Period. Math. Hung. 2023].

Theorem 7.1.1. (Slippery slope metric)

*Let a slippery slope of a mountain (M, h) be an n-dimensional Riemannian manifold, $n > 1$, with the gravitational wind $\mathbf{G}^T$ on M and the cross-traction coefficient $\eta \in [0, 1]$. The time-minimal paths on (M, h) in the presence of an active wind $\mathbf{G}_\eta$ as in (38) are the geodesics of the **slippery slope metric** $\tilde{F}_\eta$ which satisfies*

$$\tilde{F}_\eta \sqrt{\alpha^2 + 2(1-\eta)\bar{g}\beta\tilde{F}_\eta + (1-\eta)^2 ||\mathbf{G}^T||_h^2 \tilde{F}_\eta^2} = \alpha^2 + (2-\eta)\bar{g}\beta\tilde{F}_\eta + (1-\eta)||\mathbf{G}^T||_h^2 \tilde{F}_\eta^2$$

with $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ given by (39), where either $\eta \in [0, \frac{1}{2}]$ and $||\mathbf{G}^T||_h < 1$, or $\eta \in (\frac{1}{2}, 1]$ and $||\mathbf{G}^T||_h < \frac{1}{2\eta}$. In particular, if $\eta = 1$, then the slippery slope metric is reduced to the Matsumoto metric, and if $\eta = 0$, then it is the Randers metric which solves the Zermelo navigation problem on a Riemannian manifold under a gravitational wind $\mathbf{G}^T$.

Theorem 7.1.2. (Time geodesics)

Let a slippery slope of a mountain (M, h) *be an* n*-dimensional Riemannian manifold,* $n > 1$*, with the gravitational wind* $\mathbf{G}^T$ *and the cross-traction coefficient* $\eta \in [0, 1]$*. The* ***time-minimal paths*** *on* (M, h) *in the presence of an active wind* $\mathbf{G}_\eta$ *as in* (38) *are the time-parametrized solutions* $\gamma(t) = (\gamma^i(t))$*,* $i = 1, ..., n$ *of the ODE system*

$$\ddot{\gamma}^i(t) + 2\tilde{\mathcal{G}}^i_\eta(\gamma(t), \dot{\gamma}(t)) = 0, \tag{40}$$

for each $\eta \in [0, 1]$*, where*

$$\begin{aligned} \tilde{\mathcal{G}}^i_\eta(\gamma(t), \dot{\gamma}(t)) \quad = \quad & \mathcal{G}^i_\alpha(\gamma(t), \dot{\gamma}(t)) + \left[\tilde{\Theta}(r_{00} + 2\alpha^2 \tilde{R} r) + \alpha \tilde{\Omega} r_0\right] \frac{\dot{\gamma}^i(t)}{\alpha} \\ & - \left[\tilde{\Psi}(r_{00} + 2\alpha^2 \tilde{R} r) + \alpha \tilde{\Pi} r_0\right] \frac{w^i}{\bar{g}} - \tilde{R} {w^i}_{|j} \frac{\alpha^2 w^j}{\bar{g}^2}, \end{aligned}$$

with

Theorem 7.1.2. (Time geodesics) - cont.

$$\mathcal{G}^i_\alpha(\gamma(t),\dot\gamma(t)) = \frac{1}{4}h^{im}\left(2\frac{\partial h_{jm}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^m}\right)\dot\gamma^j(t)\dot\gamma^k(t),$$

$$r_{00} = -\frac{1}{\bar g}w_{j|k}\dot\gamma^j(t)\dot\gamma^k(t), \qquad r_0 = \frac{1}{\bar g^2}w_{j|k}\dot\gamma^j(t)w^k, \qquad r = -\frac{1}{\bar g^3}w_{j|k}w^j w^k,$$

$$\tilde R = \frac{(1-\eta)\bar g^2}{2\tilde B\alpha^4}(\tilde B\alpha^2 + 2\eta), \qquad \tilde\Theta = \frac{\bar g^2\alpha(\tilde A\tilde B^2\alpha^2 - 2\tilde D^2\beta)}{2\tilde E}, \qquad \tilde\Psi = \frac{\bar g^2\alpha^2(\tilde A^2\tilde B + 2\tilde D^2)}{2\tilde E},$$

$$\tilde\Omega = \frac{(1-\eta)\bar g^2}{\tilde B\tilde E\alpha^2}[(\tilde B\alpha^2 + 2\eta)(\bar g^2\tilde B^3\alpha^2 + 2\tilde D^2||\mathbf{G}^T||^2_h) - 4\eta\tilde D(\bar g^2\tilde B\beta + \tilde A||\mathbf{G}^T||^2_h)],$$

$$\tilde\Pi = \frac{(1-\eta)\bar g^4}{\tilde B\tilde E\alpha^3}[4\eta\tilde C\tilde D\alpha + (\tilde B\alpha^2 + 2\eta)(\tilde A\tilde B^2\alpha^2 - 2\tilde D^2\beta)],$$

$$\tilde A = -\frac{2\bar g}{\alpha^2}\left\{(1-\eta)[1-(2-\eta)||\mathbf{G}^T||^2_h] - (2-\eta)^2\bar g\beta - (2-\eta)\alpha^2\right\},$$

$$\tilde B = -\frac{2}{\alpha^2}\left\{[1-2(1-\eta)||\mathbf{G}^T||^2_h] - 2(2-\eta)\bar g\beta - 2\alpha^2\right\}, \qquad \tilde C = \frac{1}{\alpha}\left(\tilde B\alpha^2 + \tilde A\beta\right),$$

$$\tilde D = 2\tilde A - (2-\eta)\bar g\tilde B, \qquad \tilde E = \bar g^2\tilde B\tilde C^2\alpha^2 + (||\mathbf{G}^T||^2_h\alpha^2 - \bar g^2\beta^2)(\tilde A^2\tilde B + 2\tilde D^2)$$

and $\alpha = \alpha(\gamma(t),\dot\gamma(t))$, $\beta = \beta(\gamma(t),\dot\gamma(t))$.

8.1 Model of a slope under the cross-gravity effect $\rightsquigarrow$ Cross slope

N. Aldea, P. Kopacz, The slope-of-a-mountain problem in a cross gravitational wind. *Nonlinear Anal.-Theor.*, 235 (2023), 113329.

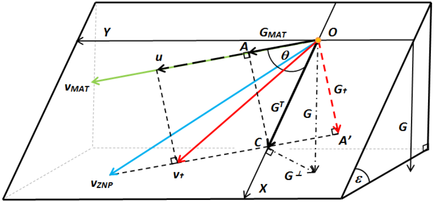
N. Aldea, P. Kopacz, Time geodesics on a slippery cross slope under gravitational wind. *Nonlinear Anal. Real World Appl.*, 81 (2025), 104177.

$$\mathbf{G}^T = \underbrace{\text{Proj}_u \mathbf{G}^T}_{\substack{\mathbf{G}_{MAT} \\ \text{along-gravity effect}}} + \underbrace{\text{Proj}_{u^\perp} \mathbf{G}^T}_{\substack{\mathbf{G}^\perp{}_{MAT} \\ \text{cross-gravity effect}}} \quad \rightsquigarrow \quad \text{if along-gravity effect is omitted}$$

$\rightsquigarrow$ Cross slope (CROSS) $\rightsquigarrow$ impact of $\mathbf{G}^T$-components is reversed in comparison to MAT

active wind $\rightsquigarrow$ cross gravitational wind $\quad \mathbf{G}_\dagger = \mathbf{G}^\perp{}_{MAT} = -\mathbf{G}_{MAT} + \mathbf{G}^T \quad (41)$

- **Resultant velocity:** $v_\dagger = u + \text{Proj}_{u^\perp} \mathbf{G}^T = u + \mathbf{G}_\dagger$.



8.1.2 Main results $\rightsquigarrow$ Cross-slope metric

CROSS problem: *Suppose a person walks on a horizontal plane at a constant speed, while the gravity acts orthogonally to this plane. Imagine the person endeavours to walk now on a slope of a mountain under the influence of a cross gravitational wind.*

How should the person navigate on the slope to get from one point to another in the shortest time?

Theorem 8.1.1. (Cross-slope metric)

*Let the slope of a mountain be an n-dimensional Riemannian manifold (M, h), $n > 1$, with the gravitational wind $\mathbf{G}^T$. The time-minimal paths on (M, h) in the presence of the cross gravitational wind $\mathbf{G}_\dagger$ as in (41) are the geodesics of the **cross-slope metric** F which satisfies*

$$||\mathbf{G}^T||_h^2 F^4 + 2\bar{g}\beta F^3 + (\alpha^2 - \bar{g}^2\beta^2)F^2 - 2\bar{g}\alpha^2\beta F - \alpha^4 = 0, \tag{42}$$

where $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ are given by (39) and $||\mathbf{G}^T||_h < \frac{1}{2}$.

8.1.2 Main results ⇝ Time geodesics

Theorem 8.1.2. (Time geodesics)

*Let the slope of a mountain be an n-dimensional Riemannian manifold (M,h), $n>1$, with the gravitational wind $\mathbf{G}^T$. The **time-minimal paths** on (M,h) in the presence of the cross wind $\mathbf{G}_\dagger$ are the time-parametrized solutions $\gamma(t)=(\gamma^i(t))$, $i=1,...,n$ of the ODE system*

$$\ddot{\gamma}^i(t)+2\mathcal{G}^i(\gamma(t),\dot{\gamma}(t))=0, \tag{43}$$

for each $\eta\in[0,1]$, where

$$\begin{aligned}\mathcal{G}^i(\gamma(t),\dot{\gamma}(t)) &= \mathcal{G}^i_\alpha(\gamma(t),\dot{\gamma}(t))+\left[\tilde{\Theta}(r_{00}+2\alpha^2\tilde{R}r)+\alpha\tilde{\Omega}r_0\right]\frac{\dot{\gamma}^i(t)}{\alpha}\\ &\quad -\left[\tilde{\Psi}(r_{00}+2\alpha^2\tilde{R}r)+\alpha\tilde{\Pi}r_0\right]\frac{w^i}{\bar{g}}-\tilde{R}w^i{}_{|j}\frac{\alpha^2w^j}{\bar{g}^2},\end{aligned}$$

with

$$\begin{aligned}&\mathcal{G}^i_\alpha(\gamma(t),\dot{\gamma}(t))=\tfrac{1}{4}h^{im}\left(2\frac{\partial h_{jm}}{\partial\gamma^k}-\frac{\partial h_{jk}}{\partial\gamma^m}\right)\dot{\gamma}^j(t)\dot{\gamma}^k(t), \qquad \tilde{R}=-\frac{\bar{g}^2}{2\alpha^4\tilde{B}},\\ &r_{00}=-\frac{1}{\bar{g}}w_{j|k}\dot{\gamma}^j(t)\dot{\gamma}^k(t), \qquad r_0=\frac{1}{\bar{g}^2}w_{j|k}\dot{\gamma}^j(t)w^k, \qquad r=-\frac{1}{\bar{g}^3}w_{j|k}w^jw^k,\\ &\tilde{\Theta}=\frac{\bar{g}\alpha}{2\tilde{E}}(\alpha^6\tilde{A}\tilde{B}^2-\bar{g}\beta), \qquad \tilde{\Omega}=-\frac{\bar{g}^2}{\tilde{B}\tilde{E}}[\alpha^4\tilde{B}^3+\bar{g}\beta\tilde{B}+||\mathbf{G}^T||_h^2(\tilde{B}-\tilde{A})],\\ &\tilde{\Psi}=\frac{\bar{g}^2\alpha^2}{2\tilde{E}}(\alpha^4\tilde{A}^2\tilde{B}+1), \qquad \tilde{\Pi}=-\frac{\bar{g}^3}{2\tilde{B}\tilde{E}\alpha^3}[2\alpha^4\tilde{B}(\alpha^2\tilde{A}\tilde{B}-1)-\bar{g}\beta(\alpha^2\tilde{B}+1)],\\ &\tilde{A}=\frac{1}{\alpha^2}(\bar{g}\beta+\alpha^2-1), \qquad \tilde{B}=\frac{1}{\alpha^2}(2\bar{g}\beta+2\alpha^2-1), \quad \tilde{C}=\frac{1}{\alpha}\left(\alpha^2\tilde{B}+\bar{g}\beta\tilde{A}\right),\\ &\tilde{E}=\tilde{B}\tilde{C}^2\alpha^6+(||\mathbf{G}^T||_h^2\alpha^2-\bar{g}^2\beta^2)(\alpha^4\tilde{A}^2\tilde{B}+1)\end{aligned} \tag{44}$$

and $\alpha=\alpha(\gamma(t),\dot{\gamma}(t))$, $\beta=\beta(\gamma(t),\dot{\gamma}(t))$.

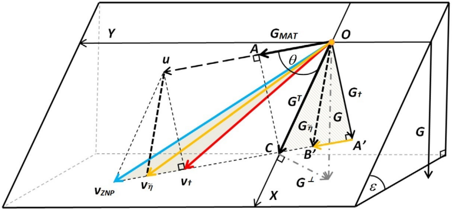
8.3 Model of a slippery cross slope under gravitational wind

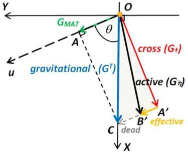
$$\mathbf{G}^T = \underbrace{\text{Proj}_u \mathbf{G}^T}_{\substack{\mathbf{G}_{MAT} \\ \text{along-gravity effect}}} + \underbrace{\text{Proj}_{u^\perp} \mathbf{G}^T}_{\substack{\mathbf{G}^\perp{}_{MAT} \\ \text{cross-gravity effect}}}$$

- along-gravity effect is partially cancelled $\rightsquigarrow$ along-traction coefficient $\tilde{\eta} \in [0, 1]$
- $\rightsquigarrow$ **Slippery cross slope** (S-CROSS)
- $\rightsquigarrow$ **active wind** $\mathbf{G}_{\tilde{\eta}} = (1 - \tilde{\eta})\text{Proj}_u \mathbf{G}^T + \text{Proj}_{u^\perp} \mathbf{G}^T$

$$\Rightarrow \quad \mathbf{G}_{\tilde{\eta}} = \underbrace{-\tilde{\eta}\, \mathbf{G}_{MAT}}_{\text{I. } u\text{-direction-dependent deformation}} + \underbrace{\mathbf{G}^T}_{\text{II. rigid translation}} \qquad (45)$$

- **Resultant velocity:** $v_{\tilde{\eta}} = u + \mathbf{G}_{\tilde{\eta}}$.





8.3 Model of a slippery cross slope under $\mathbf{G}^T \rightsquigarrow$ Slippery-cross-slope metric

S-CROSS problem: *Suppose a craft or a vehicle goes on a horizontal plane at maximum constant speed, while gravity acts perpendicularly on this plane. Imagine the craft moves now on a slippery cross slope of a mountain, with a given along-traction coefficient and under gravity.*

What path should be followed by the craft to get from one point to another in the minimum time?

Theorem 8.3.1. (Slippery-cross-slope metric)

*Let a slippery cross slope of a mountain be an n-dimensional Riemannian manifold (M, h), $n > 1$, with the along-traction coefficient $\tilde{\eta} \in [0, 1]$ and the gravitational wind $\mathbf{G}^T$ on M. The time-minimal paths on (M, h) under the action of an active wind $\mathbf{G}_{\tilde{\eta}}$ as in (45) are the geodesics of the **slippery-cross-slope metric** $\tilde{F}_{\tilde{\eta}}$ which satisfies*

$$\tilde{F}_{\tilde{\eta}}\sqrt{\alpha^2 + 2\bar{g}\beta\tilde{F}_{\tilde{\eta}} + ||\mathbf{G}^T||_h^2\tilde{F}_{\tilde{\eta}}^2} = \alpha^2 + (2-\tilde{\eta})\bar{g}\beta\tilde{F}_{\tilde{\eta}} + (1-\tilde{\eta})||\mathbf{G}^T||_h^2\tilde{F}_{\tilde{\eta}}^2,$$

with $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ given by (39), where either $\tilde{\eta} \in [0, \frac{1}{3}]$ and $||\mathbf{G}^T||_h < \frac{1}{1-\tilde{\eta}}$, or $\tilde{\eta} \in (\frac{1}{3}, 1]$ and $||\mathbf{G}^T||_h < \frac{1}{2\tilde{\eta}}$. In particular, if $\tilde{\eta} = 1$, then the slippery-cross-slope metric yields the cross-slope metric, and if $\tilde{\eta} = 0$, then it is the Randers metric which solves the Zermelo navigation problem on a Riemannian manifold under a gravitational wind $\mathbf{G}^T$.

8.3 Model of a slippery cross slope under $\mathbf{G}^T \rightsquigarrow$ Time geodesics

Theorem 8.3.2. (Time geodesics)

*Let a slippery cross slope of a mountain be an n-dimensional Riemannian manifold (M, h), $n > 1$, with the along-traction coefficient $\tilde{\eta} \in [0, 1]$ and the gravitational wind $\mathbf{G}^T$ on M. The **time-minimal paths** on (M, h) under the action of an active wind $\mathbf{G}_{\tilde{\eta}}$ as in* (45) *are the time-parametrized solutions $\gamma(t) = (\gamma^i(t))$, $i = 1, ..., n$ of the ODE system*

$$\ddot{\gamma}^i(t) + 2\tilde{\mathcal{G}}^i_{\tilde{\eta}}(\gamma(t), \dot{\gamma}(t)) = 0, \qquad (46)$$

for each $\eta \in [0, 1]$, where

$$\tilde{\mathcal{G}}^i_{\tilde{\eta}}(\gamma(t), \dot{\gamma}(t)) = \mathcal{G}^i_{\alpha}(\gamma(t), \dot{\gamma}(t)) + \left[\tilde{\Theta}(r_{00} + 2\alpha^2 \tilde{R} r) + \alpha \tilde{\Omega} r_0\right] \frac{\dot{\gamma}^i(t)}{\alpha} - \left[\tilde{\Psi}(r_{00} + 2\alpha^2 \tilde{R} r) + \alpha \tilde{\Pi} r_0\right] \frac{w^i}{\bar{g}} - \tilde{R} {w^i}_{|j} \frac{\alpha^2 w^j}{\bar{g}^2},$$

with

8.3 Model of a slippery cross slope under $\mathbf{G}^T \rightsquigarrow$ Time geodesics

Theorem 8.3.2. (Time geodesics) - cont.

$$\mathcal{G}^i_\alpha(\gamma(t),\dot{\gamma}(t)) = \frac{1}{4}h^{im}\left(2\frac{\partial h_{jm}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^m}\right)\dot{\gamma}^j(t)\dot{\gamma}^k(t), \qquad \tilde{\Psi} = \frac{\bar{g}^2\alpha^2}{2\tilde{E}}(\alpha^4\tilde{A}^2\tilde{B} + \tilde{\eta}^2),$$

$$r_{00} = -\frac{1}{\bar{g}}w_{j|k}\dot{\gamma}^j(t)\dot{\gamma}^k(t), \qquad r_0 = \frac{1}{\bar{g}^2}w_{j|k}\dot{\gamma}^j(t)w^k, \qquad r = -\frac{1}{\bar{g}^3}w_{j|k}w^j w^k,$$

$$\tilde{R} = \frac{\bar{g}^2}{2\alpha^4\tilde{B}}[(1-\tilde{\eta})\,\alpha^2\tilde{B} - \tilde{\eta}], \qquad \tilde{\Theta} = \frac{\bar{g}\alpha}{2\tilde{E}}(\alpha^6\tilde{A}\tilde{B}^2 - \tilde{\eta}^2\bar{g}\beta),$$

$$\tilde{\Omega} = \frac{\bar{g}^2}{\alpha^2\tilde{B}\tilde{E}}\{[(1-\tilde{\eta})\,\alpha^2\tilde{B} - \tilde{\eta}](\alpha^6\tilde{B}^3 + \tilde{\eta}^2||\mathbf{G}^T||_h^2) - \tilde{\eta}^2\alpha^2(\bar{g}\beta\tilde{B} + ||\mathbf{G}^T||_h^2\tilde{A})\},$$

$$\tilde{\Pi} = \frac{\bar{g}^3}{2\alpha^3\tilde{B}\tilde{E}}\{[(1-\tilde{\eta})\,\alpha^2\tilde{B} - \tilde{\eta}](2\alpha^6\tilde{A}\tilde{B}^2 - \tilde{\eta}^2\bar{g}\beta) + \tilde{\eta}^2\alpha^2\tilde{B}(2\alpha^2 + \bar{g}\beta)\},$$

$$\tilde{A} = -\frac{1}{\alpha^2}\{\left[1 - (2-\tilde{\eta})\,(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right] - (2-\tilde{\eta})^2\bar{g}\beta - (2-\tilde{\eta})\alpha^2\},$$

$$\tilde{B} = -\frac{1}{\alpha^2}\{[1 - 2(1-\tilde{\eta})||\mathbf{G}^T||_h^2] - 2(2-\tilde{\eta})\bar{g}\beta - 2\alpha^2\},$$

$$\tilde{C} = \frac{1}{\alpha}\left(\alpha^2\tilde{B} + \bar{g}\beta\tilde{A}\right), \qquad \tilde{E} = \alpha^6\tilde{B}\tilde{C}^2 + (||\mathbf{G}^T||_h^2\alpha^2 - \bar{g}^2\beta^2)(\alpha^4\tilde{A}^2\tilde{B} + \tilde{\eta}^2)$$

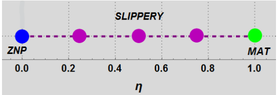
and $\alpha = \alpha(\gamma(t),\dot{\gamma}(t))$, $\beta = \beta(\gamma(t),\dot{\gamma}(t))$.

9.1 Broader meaning of a slippery slope $\rightsquigarrow$ General model $\sim$ 2-parameter model

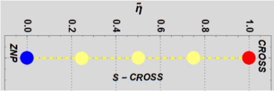
N. Aldea, P. Kopacz, A general model for time-minimizing navigation on a mountain slope under gravity.
J. Geom. Anal., 35 (2025) 282.

1-parameter models:

1. SLIPPERY model: $\eta \in [0, 1] \rightsquigarrow$ particular cases MAT, ZNP and many other

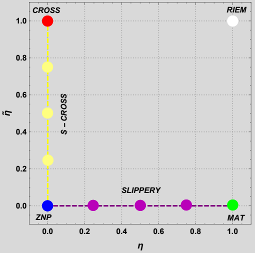


2. S-CROSS model: $\tilde{\eta} \in [0, 1] \rightsquigarrow$ particular cases CROSS, ZNP and many other

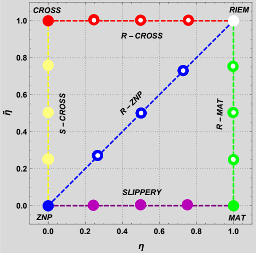


$\rightsquigarrow$ ZNP is border for SLIPPERY and S-CROSS: $\eta = \tilde{\eta} = 0$

9.1 Broader meaning of a slippery slope $\rightsquigarrow$ General model $\sim$ 2-parameter model



$\rightsquigarrow$



$\rightsquigarrow$ MAT, ZNP, RIEM, CROSS $\rightsquigarrow$ corner and particular cases

$\rightsquigarrow$ **other 1-parameter models:**

- R-MAT - *reduced Matsumoto slope-of-a-mountain problem* ($\eta = 1$ & $\tilde{\eta} \in [0,1]$)

$$v = u + (1-\tilde{\eta})\mathbf{G}_{MAT} \rightsquigarrow \tilde{F}(x,y) = \frac{||y||_h^2}{||y||_h + (1-\tilde{\eta})h(y,\mathbf{G}^T)}, \quad ||\mathbf{G}^T||_h < \frac{1}{2(1-\tilde{\eta})}.$$

- R-ZNP - *reduced Zermelo navigation problem* ($\eta = \tilde{\eta} \in [0,1]$)

$$v = u + (1-\eta)\mathbf{G}^T \rightsquigarrow \tilde{F}(x,y) = \frac{\sqrt{[(1-\eta)h(y,\mathbf{G}^T)]^2 + \lambda_\eta ||y||_h^2}}{\lambda_\eta} - \frac{(1-\eta)h(y,\mathbf{G}^T)}{\lambda_\eta},$$

with $||\mathbf{G}^T||_h < \frac{1}{1-\eta}$ and $\lambda_\eta = 1-(1-\eta)^2||\mathbf{G}^T||_h^2$.

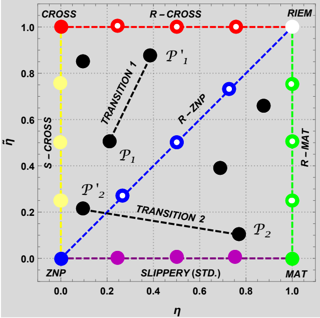
9.1 Broader meaning of a slippery slope $\rightsquigarrow$ General model $\sim$ 2-parameter model

- <u>R-CROSS</u> - *reduced cross slope problem* $(\eta \in [0,1]\ \&\ \tilde{\eta} = 1)$

$$v = u + (1-\eta)\mathbf{G}^{\perp}_{MAT} \rightsquigarrow \boxed{\tilde{F}(x,y) = \ldots ☹}, \quad ||\mathbf{G}^T||_h < \frac{1}{2(1-\eta)}.$$

- Varying simultaneously both $\eta, \tilde{\eta} \in [0,1] \rightsquigarrow$ 2-parameter model of slippery mountain slope under the action of gravity $\sim$ **General model**

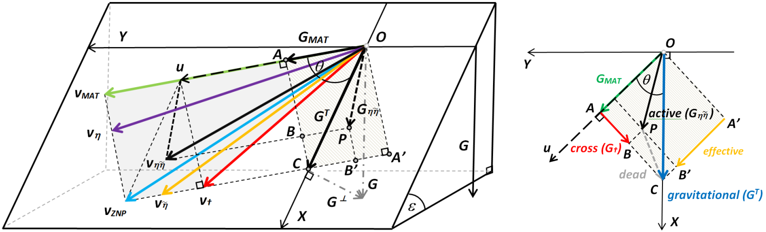
  $\rightsquigarrow \tilde{\mathcal{S}} = [0,1] \times [0,1]$ - **complete problem square diagram** including: all navigation problems $\mathcal{P}_{\eta,\tilde{\eta}}$ on the slippery slope under gravity, with $(\eta,\tilde{\eta}) \in \mathcal{S} = \tilde{\mathcal{S}} \setminus \{(1,1)\}$ and transitions $\mathcal{T}_{\mathcal{P}}^{\mathcal{P}'}$ between $\mathcal{P}_{\eta,\tilde{\eta}}$ and $\mathcal{P}'_{\eta',\tilde{\eta}'}$.



9.1.2 Problem formulation and main theorems

GENERAL problem: *Suppose a walker, craft or a vehicle has a certain constant maximum speed as measured on a horizontal plane, while gravity acts perpendicular to this plane. Imagine now that the craft endeavours to move on a slippery slope of a mountain under gravity, admitting a traction-dependent sliding in arbitrary (downward) direction.*

What path should be followed by the craft to get from one point to another in the least time?



- **Active wind:** $\mathbf{G}_{\eta\tilde{\eta}} = (1-\tilde{\eta})\text{Proj}_u \mathbf{G}^T + (1-\eta)\text{Proj}_{u^\perp} \mathbf{G}^T$, $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}}$

$$\Rightarrow \quad \mathbf{G}_{\eta\tilde{\eta}} = \underbrace{(\eta-\tilde{\eta})\mathbf{G}_{MAT}}_{\text{I. anisotropic deformation}} + \underbrace{(1-\eta)\,\mathbf{G}^T}_{\text{II. rigid translation}} \qquad (47)$$

- **Resultant velocity:** $v_{\eta\tilde{\eta}} = u + \mathbf{G}_{\eta\tilde{\eta}}$, $(\eta, \tilde{\eta}) \in \tilde{\mathcal{S}}$.

9.1.2 Problem formulation and main theorems $\rightsquigarrow$ $(\eta,\tilde{\eta})$-slope metric

Theorem 9.1.1. ($(\eta,\tilde{\eta})$-slope metric)

*Let a slippery slope of a mountain be an n-dimensional Riemannian manifold (M,h), $n>1$, with a cross-traction coefficient $\eta\in[0,1]$, an along-traction coefficient $\tilde{\eta}\in[0,1]$ and a gravitational wind $\mathbf{G}^T$ on M. The time-minimal paths on (M,h) under the action of an active wind $\mathbf{G}_{\eta\tilde{\eta}}$ as in (47) are the geodesics of an $(\eta,\tilde{\eta})$-**slope metric** $\tilde{F}_{\eta\tilde{\eta}}$, which satisfies*

$$\tilde{F}_{\eta\tilde{\eta}}\sqrt{\alpha^2+2(1-\eta)\bar{g}\beta\tilde{F}_{\eta\tilde{\eta}}+(1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}_{\eta\tilde{\eta}}^2}=\alpha^2+(2-\eta-\tilde{\eta})\bar{g}\beta\tilde{F}_{\eta\tilde{\eta}}+(1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2\tilde{F}_{\eta\tilde{\eta}}^2$$

with $\alpha=\alpha(x,y)$, $\beta=\beta(x,y)$ given by (39), where either $||\mathbf{G}^T||_h<\frac{1}{1-\tilde{\eta}}$ and $(\eta,\tilde{\eta})\in\mathcal{D}_1\cup\mathcal{D}_2$, or $||\mathbf{G}^T||_h<\frac{1}{2|\eta-\tilde{\eta}|}$ and $(\eta,\tilde{\eta})\in\mathcal{D}_3\cup\mathcal{D}_4$, where

$$\mathcal{D}_1=\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\eta\geq\tilde{\eta}>2\eta-1\},\qquad\mathcal{D}_2=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\frac{3\tilde{\eta}-1}{2}<\eta<\tilde{\eta}\right\},$$

$$\mathcal{D}_3=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\eta\geq\frac{1}{2},\ \tilde{\eta}\leq2\eta-1\right\},\qquad\mathcal{D}_4=\left\{(\eta,\tilde{\eta})\in\mathcal{S}\mid\tilde{\eta}\geq\frac{1}{3},\ \eta\leq\frac{3\tilde{\eta}-1}{2}\right\},$$

$\mathcal{S}=\bigcup_{i=1}^{4}\mathcal{D}_i$ and $\mathcal{D}_i\cap\mathcal{D}_j=\varnothing$, for any $i\neq j$, $i,j=1,...,4$. No restriction should be imposed on $||\mathbf{G}^T||_h$ if $\eta=\tilde{\eta}=1$. In particular, a slope metric of type $(0,0)$, $(1,0)$ $(0,1)$, $(1,1)$ is reduced to a Randers metric, a Matsumoto metric, a cross slope metric and a Riemannian metric h, respectively.

9.1.2 Problem formulation and main theorems ⇝ Time geodesics

Theorem 9.1.2. (Time geodesics)

*Let a slippery slope of a mountain be an n-dimensional Riemannian manifold (M, h), $n > 1$, with a cross-traction coefficient $\eta \in [0, 1]$, an along-traction coefficient $\tilde{\eta} \in [0, 1]$ and a gravitational wind $\mathbf{G}^T$ on M. The **time-minimal paths** on (M, h) under the action of an active wind $\mathbf{G}_{\eta\tilde{\eta}}$ as in* (47) *are the time-parametrized solutions $\gamma(t) = (\gamma^i(t))$, $i = 1, ..., n$ of the ODE system*

$$\ddot{\gamma}^i(t) + 2\tilde{\mathcal{G}}^i_{\eta\tilde{\eta}}(\gamma(t), \dot{\gamma}(t)) = 0, \tag{48}$$

where

$$\begin{aligned} \tilde{\mathcal{G}}^i_{\eta\tilde{\eta}}(\gamma(t), \dot{\gamma}(t)) &= \mathcal{G}^i_{\alpha}(\gamma(t), \dot{\gamma}(t)) + \left[\tilde{\Theta}(r_{00} + 2\alpha^2 \tilde{R} r) + \alpha\tilde{\Omega} r_0\right] \frac{\dot{\gamma}^i(t)}{\alpha} \\ &\quad - \left[\tilde{\Psi}(r_{00} + 2\alpha^2 \tilde{R} r) + \alpha\tilde{\Pi} r_0\right] \frac{w^i}{\bar{g}} - \tilde{R} w^i{}_{|j} \frac{\alpha^2 w^j}{\bar{g}^2} \end{aligned}$$

with

9.1.2 Problem formulation and main theorems $\rightsquigarrow$ Time geodesics

Theorem 9.1.2. (Time geodesics) - cont.

$$\mathcal{G}^i_\alpha(\gamma(t),\dot\gamma(t)) = \frac{1}{4}h^{im}\left(2\frac{\partial h_{jm}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^m}\right)\dot\gamma^j(t)\dot\gamma^k(t), \qquad \tilde\Psi = \frac{\bar g^2\alpha^2}{2\tilde E}[\alpha^4\tilde A^2\tilde B + (\tilde\eta-\eta)^2],$$

$$r_{00} = -\frac{1}{\bar g}w_{j|k}\dot\gamma^j(t)\dot\gamma^k(t), \qquad r_0 = \frac{1}{\bar g^2}w_{j|k}\dot\gamma^j(t)w^k, \qquad r = -\frac{1}{\bar g^3}w_{j|k}w^jw^k,$$

$$\tilde R = \frac{(1-\eta)\bar g^2}{2\alpha^4\tilde B}[(1-\tilde\eta)\,\alpha^2\tilde B - (\tilde\eta-\eta)], \qquad \tilde\Theta = \frac{\bar g\alpha}{2\tilde E}[\alpha^6\tilde A\tilde B^2 - (\tilde\eta-\eta)^2\bar g\beta],$$

$$\tilde\Omega = \frac{(1-\eta)\bar g^2}{\alpha^2\tilde B\tilde E}\{[(1-\tilde\eta)\,\alpha^2\tilde B - (\tilde\eta-\eta)][\alpha^6\tilde B^3 + (\tilde\eta-\eta)^2||\mathbf{G}^T||_h^2] - (\tilde\eta-\eta)^2\alpha^2(\bar g\beta\tilde B + ||\mathbf{G}^T||_h^2\tilde A)\},$$

$$\tilde\Pi = \frac{(1-\eta)\bar g^3}{2\alpha^3\tilde B\tilde E}\{[(1-\tilde\eta)\,\alpha^2\tilde B - (\tilde\eta-\eta)][2\alpha^6\tilde A\tilde B^2 - (\tilde\eta-\eta)^2\bar g\beta] + (\tilde\eta-\eta)^2\alpha^2\tilde B[2\alpha^2 + (1-\eta)\bar g\beta]\},$$

$$\tilde A = -\frac{1}{\alpha^2}\{(1-\eta)\left[1-(2-\eta-\tilde\eta)\,(1-\tilde\eta)\,||\mathbf{G}^T||_h^2\right] - (2-\eta-\tilde\eta)^2\bar g\beta - (2-\eta-\tilde\eta)\alpha^2\},$$

$$\tilde B = -\frac{1}{\alpha^2}\{[1-2(1-\eta)(1-\tilde\eta)||\mathbf{G}^T||_h^2] - 2(2-\eta-\tilde\eta)\bar g\beta - 2\alpha^2\},$$

$$\tilde C = \frac{1}{\alpha}\left(\alpha^2\tilde B + \bar g\beta\tilde A\right), \qquad \tilde E = \alpha^6\tilde B\tilde C^2 + (||\mathbf{G}^T||_h^2\alpha^2 - \bar g^2\beta^2)[\alpha^4\tilde A^2\tilde B + (\tilde\eta-\eta)^2]$$

and $\alpha = \alpha(\gamma(t),\dot\gamma(t))$, $\beta = \beta(\gamma(t),\dot\gamma(t))$, *and* w^i *denoting the components of* $\mathbf{G}^T$.

9.3 Proofs of the main results $\rightsquigarrow$ Sketches

Theorem 9.1.1. - sketch of proof $\rightsquigarrow$ Step I

$\mathcal{P}_{\eta,\tilde{\eta}}$ on (M, h) under

$$\mathbf{G}_{\eta\tilde{\eta}} = \underbrace{(\eta-\tilde{\eta})\mathbf{G}_{MAT}}_{\text{Step I (anisotropic deformation)}} + \underbrace{(1-\eta)\,\mathbf{G}^T}_{\text{Step II (rigid translation)}} \rightsquigarrow v_{\eta\tilde{\eta}} = u + \mathbf{G}_{\eta\tilde{\eta}}$$

$F(x, -(1-\eta)\mathbf{G}^T) < 1$

Step I: deformation of h by $(\eta-\tilde{\eta})\mathbf{G}_{MAT} \rightsquigarrow v = u + (\eta-\tilde{\eta})\mathbf{G}_{MAT}$, $\forall\ (\eta,\tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}$, $\mathcal{L} = \{(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}} \mid \eta = \tilde{\eta}\}$; $\eta = \tilde{\eta} \Rightarrow v = u$.

C1: $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h < 1$; **C2:** $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h = 1$; **C3:** $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h > 1 \Rightarrow$

- under $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h < 1$ (performed for any direction), the deformation of h by $(\eta-\tilde{\eta})\mathbf{G}_{MAT} \Rightarrow$ a Finsler metric of Matsumoto type

$$F(x,y) = \frac{\alpha^2}{\alpha - (\eta-\tilde{\eta})\bar{g}\beta} \quad \text{iff} \quad ||\mathbf{G}^T||_h < \frac{1}{2|\eta-\tilde{\eta}|}, \quad \forall(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}} \setminus \mathcal{L}.$$

- when $|\eta-\tilde{\eta}|\ ||\mathbf{G}_{MAT}||_h \geq 1$ (only for some directions), this deformation $\not\Rightarrow$ a Finsler metric.

9.3 Proofs of the main results $\rightsquigarrow$ Sketches

Theorem 9.1.1. - sketch of proof $\rightsquigarrow$ Step II

Step II: exploring Zermelo's navigation on (M, F) with the navig. data $(F, (1-\eta)\mathbf{G}^T)$, $F(x,y) = \frac{\alpha^2}{\alpha-(\eta-\tilde{\eta})\bar{g}\beta}$, $\forall(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}} \Rightarrow$ $(\eta,\tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$ and the necessary and sufficient conditions that its indicatrix $I_{\tilde{F}_{\eta\tilde{\eta}}}$ is strongly convex.

- Applying Proposition [Z. Shen, Canad. J. Math. 2003], for each $(\eta,\tilde{\eta}) \in \tilde{\mathcal{S}}$, the $(\eta,\tilde{\eta})$-slope metric is the unique positive solution $\tilde{F}$ of the irrational eq.

$$\tilde{F}\sqrt{\alpha^2 + 2(1-\eta)\bar{g}\beta\tilde{F} + (1-\eta)^2||\mathbf{G}^T||_h^2\tilde{F}^2} = \alpha^2 + (2-\eta-\tilde{\eta})\bar{g}\beta\tilde{F} + (1-\eta)(1-\tilde{\eta})||\mathbf{G}^T||_h^2\tilde{F}^2 \qquad (49)$$

which is equivalent to

$$(1-\eta)^2||\mathbf{G}^T||_h^2[1-(1-\tilde{\eta})^2\,||\mathbf{G}^T||_h^2]\tilde{F}^4 + 2(1-\eta)\left[1-(2-\eta-\tilde{\eta})\,(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right]\bar{g}\beta\tilde{F}^3$$
$$+\{\left[1-2(1-\eta)\,(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right]\alpha^2 - (2-\eta-\tilde{\eta})^2\,\bar{g}^2\beta^2\}\tilde{F}^2 - 2\,(2-\eta-\tilde{\eta})\,\bar{g}\alpha^2\beta\tilde{F} - \alpha^4 = 0, \qquad (50)$$

being assumed that

$$F(x, -(1-\eta)\mathbf{G}^T) < 1. \qquad (51)$$

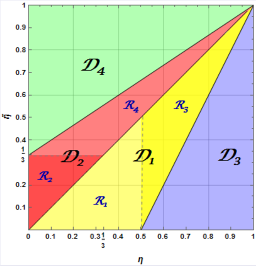
- $\tilde{F}_{\eta\tilde{\eta}}$ satisfies (49) and along any regular piecewise C^∞-curve γ, parametrized by time that represents a trajectory in Zermelo's problem, we have $\tilde{F}_{\eta\tilde{\eta}}(\gamma(t),\dot{\gamma}(t)) = 1$.

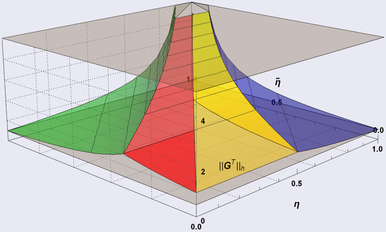
9.3 Proofs of the main results ⇝ Sketches

Theorem 9.1.1. - sketch of proof ⇝ Step II

- Exploring $F(x, -(1-\eta)\mathbf{G}^T) < 1 \Rightarrow$ the necessary and sufficient conditions that $I_{\tilde{F}_{\eta\tilde{\eta}}}$ is strongly convex can be expressed in terms of force of the gravitational wind $\mathbf{G}^T$, i.e. $||\mathbf{G}^T||_h < \tilde{b}_0$, in the problem $\mathcal{P}_{\eta,\tilde{\eta}}$, for any $(\eta, \tilde{\eta}) \in \mathcal{S}$, where

$$\tilde{b}_0 = \begin{cases} \frac{1}{1-\tilde{\eta}}, & \text{if } (\eta,\tilde{\eta}) \in \mathcal{D}_1 \cup \mathcal{D}_2 \\ \frac{1}{2|\eta-\tilde{\eta}|}, & \text{if } (\eta,\tilde{\eta}) \in \mathcal{D}_3 \cup \mathcal{D}_4 \end{cases}. \tag{52}$$





- When $(\eta, \tilde{\eta}) \in \mathcal{R}_1$, it follows that $\tilde{b}_0 \in [1, 2)$. When $(\eta, \tilde{\eta}) \in \mathcal{R}_2$, we obtain $\tilde{b}_0 \in (1, \frac{3}{2})$.
- For $(\eta, \tilde{\eta}) \in \mathcal{R}_3 \cup \mathcal{R}_4$, $\tilde{b}_0 \to \infty$ as $\tilde{\eta} \nearrow 1$. For $(\eta, \tilde{\eta}) \in \mathcal{D}_3 \cup \mathcal{D}_4$, $\tilde{b}_0 \to \infty$ as $|\eta - \tilde{\eta}| \to 0$.

9.3 Proofs of the main results $\rightsquigarrow$ Sketches

Theorem 9.1.2. - sketch of proof

- An easy argument combined with $\tilde{F}_{\eta\tilde{\eta}}$ satisfying (50) yields that $\tilde{F}_{\eta\tilde{\eta}}$ is a general (α, β)-metric,

$$\tilde{F}_{\eta\tilde{\eta}}(x,y) = \alpha\tilde{\phi}_{\eta\tilde{\eta}}(||\mathbf{G}^T||_h^2, s),$$

 with $\tilde{\phi}_{\eta\tilde{\eta}}$ a positive C^∞-function checking the identity

$$(1-\eta)^2||\mathbf{G}^T||_h^2[1-(1-\tilde{\eta})^2\,||\mathbf{G}^T||_h^2]\tilde{\phi}_{\eta\tilde{\eta}}^4 + 2(1-\eta)\left[1-(2-\eta-\tilde{\eta})\,(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2\right]\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}}^3$$
$$+[1-2(1-\eta)\,(1-\tilde{\eta})\,||\mathbf{G}^T||_h^2 - (2-\eta-\tilde{\eta})^2\,\bar{g}^2s^2]\tilde{\phi}_{\eta\tilde{\eta}}^2 - 2\,(2-\eta-\tilde{\eta})\,\bar{g}s\tilde{\phi}_{\eta\tilde{\eta}} - 1 = 0, \tag{53}$$

- Some properties regarding the function $\tilde{\phi}_{\eta\tilde{\eta}}$, implicitly given by (53), and its derivatives combined with Proposition 1 [C. Yu, H. Zhu, DGA 2011] achieve the spray coefficients of the $(\eta, \tilde{\eta})$-slope metric $\tilde{F}_{\eta\tilde{\eta}}$.
- By (36), it is immediate to supply the equations of time geodesics of $\tilde{F}_{\eta\tilde{\eta}}$.
- The argument that any time geodesic is unitary w.r.t. $\tilde{F}_{\eta\tilde{\eta}}$ because before all else, it is a trajectory in Zermelo's navigation developed in Step II, performs the proof of Theorem 9.1.2.

9.3 Proofs of the main results ⇝ Classification

A kind of **classification** of the navigation problems $\mathcal{P}_{\eta,\tilde{\eta}}$, for any $(\eta,\tilde{\eta}) \in \mathcal{S}$:

Corollary

Let $\mathcal{P}_{\eta,\tilde{\eta}}$ be a navigation problem under the action of an active wind $\mathbf{G}_{\eta\tilde{\eta}}$ given in (47), on a slippery slope of a mountain (M,h), with a cross-traction coefficient $\eta \in [0,1]$, an along-traction coefficient $\tilde{\eta} \in [0,1]$ and a gravitational wind $\mathbf{G}^T$ on M. The following statements hold:

i) For any $(\eta,\tilde{\eta}) \in \mathcal{S}$ with $\eta > \tilde{\eta}$, $\mathcal{P}_{\eta,\tilde{\eta}}$ comes from SLIPPERY with a certain form for the cross-traction coefficient, namely $c_1 = \frac{\eta-\tilde{\eta}}{1-\tilde{\eta}} \in (0,1]$;

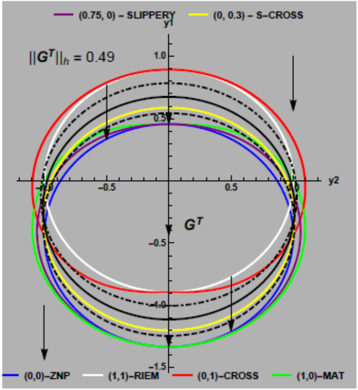
ii) For any $(\eta,\tilde{\eta}) \in \mathcal{S}$ with $\eta < \tilde{\eta}$, $\mathcal{P}_{\eta,\tilde{\eta}}$ comes from S-CROSS with a certain form for the along-traction coefficient, namely $c_2 = \frac{\tilde{\eta}-\eta}{1-\eta} \in (0,1]$.

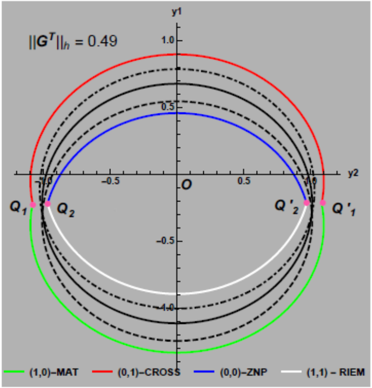
9.3 Proofs of the main results $\rightsquigarrow$ Example 1

Inclined plane: $z = x/2$ (i.e. $f(x^1, x^2) = x/2$, $x = x^1$, $y = x^2$) $\rightsquigarrow$ $||\mathbf{G}^T||_h = \frac{\bar{g}}{\sqrt{5}} = const$. Equations of indicatrix:

$$\begin{cases} -\frac{\sqrt{5}}{2}y^1 = [1 + (\eta - \tilde{\eta})\frac{\bar{g}}{\sqrt{5}}\cos\theta]\cos\theta + (1-\eta)\frac{\bar{g}}{\sqrt{5}} \\ -y^2 = [1 + (\eta - \tilde{\eta})\frac{\bar{g}}{\sqrt{5}}\cos\theta]\sin\theta \end{cases}, \quad (54)$$

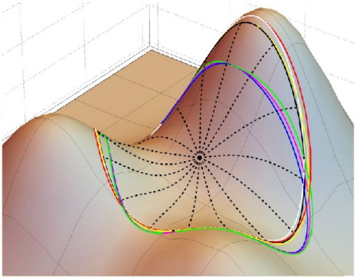
for any direction $\theta \in [0, 2\pi)$ of the velocity u.

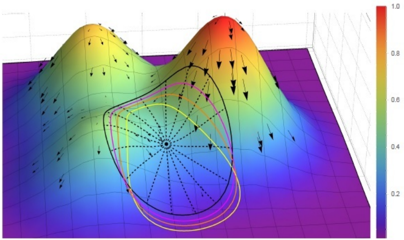




9.3 Proofs of the main results $\rightsquigarrow$ Example 2

Gaussian bell-shaped hill $\mathfrak{G}_3 : z = f(x_1, x_2) = \frac{1}{4}\sum_{k=1}^{3}(k+1)e^{-\rho_k} = \frac{1}{2}e^{-\rho_1} + \frac{3}{4}e^{-\rho_2} + e^{-\rho_3}$
$\rightsquigarrow$ with $||\mathbf{G}^T||_h \neq const.$





LEFT: On $\mathfrak{G}_3$: the time fronts centered at $(0,0)$ for the cases: MAT, ZNP, RIEM (white), SLIPPERY (with $\eta = 0.7$, magenta), S-CROSS (with $\tilde{\eta} = 0.8$, yellow), CROSS and $(0.7, 0.8)$-slope (black) and the related time geodesics (dashed colours, respectively), where $t = 2$; $\bar{g} = 0.76$.

RIGHT: The evolution of the unit time front on $\mathfrak{G}_3$ w.r.t. variable force of gravitational wind (due to changing the rescaled acceleration of gravity $\bar{g}$), where $\bar{g} \in \{$0.76 (black), 3 (magenta), 5 (orange), 7.65 (yellow)$\}$. The initial point is $(1, 0)$ and the traction coefficients are fixed, i.e. $\eta = 0.7$ and $\tilde{\eta} = 0.8$. The corresponding time geodesics in the initial setting ($\bar{g} = 0.76$) are presented in dashed black. The action of the gravitational wind is indicated by black arrows.

General view

1. Future research directions in complex Finsler geometry
2. Future research directions related to navigation problems
3. A brief list of the candidate's background $\rightsquigarrow$ Further perspectives

① Future research directions in complex Finsler geometry

1. The study of the complex Landsberg spaces

- Unicorn problem: does there exist a complex Landsberg metric (non-pure Hermitian), which is neither generalized Berwald nor Kähler?
- To investigate a new class of complex Finsler spaces (e.g. weakly Landsberg spaces), which holds $\overset{c}{L}{}^i_{jk}\eta^j = G^i_{jk}\eta^j \rightsquigarrow$ it generalizes the complex Landsberg spaces and it can be exemplified by the complex Wrona metric.

2. Projectively related complex Finsler metrics

▶ At least two directions: i. complex Finsler metrizability and ii. projective metrizability [I. Bucătaru, Z. Muzsnay, Diff. Geom. Appl. 2013 & J. Aus. Math. Soc. 2014]

i. Complex Finsler metrizability $\rightsquigarrow$ a sketch of study:

Definition

The complex spray S is complex Finsler metrizable if there exists a complex Finsler function $L = F^2$ which satisfies $S(\bar{\eta}_k) = 0$, where $\bar{\eta}_k = \frac{\partial L}{\partial \bar{\eta}^k}$. Moreover, S is weakly Kähler Finsler metrizable if it is Finsler metrizable and $S(\eta_k) = \frac{\partial L}{\partial z^k}$.

- $S(\eta_k) = \frac{\partial L}{\partial z^k}$ iff F is weakly Kähler;
- If S is weakly Kähler Finsler metrizable $\Rightarrow$ the geodesics of S are solutions of the Euler-Lagrange equations w.r.t. $L = F^2$ and S is the corresponding spray of the weakly Kähler Finsler metric F.

① Future research directions in complex Finsler geometry

- The weakly Kähler Finsler metrizability problem $\rightsquigarrow$ an inverse problem of the calculus of variation on complex manifolds restricted to weakly Kähler Finsler metrics $L \rightsquigarrow$ to find the necessary and sufficient conditions (of Helmholtz type) for the existence of two multiplier matrices $(g_{i\bar{j}}(z, \frac{dz}{dt}))$ and $(g_{ij}(z, \frac{dz}{dt}))$ such that

$$g_{i\bar{j}}(z, \frac{dz}{dt})\left(\frac{d^2\bar{z}^j}{dt^2} + 2G^{\bar{j}}(z, \frac{dz}{dt})\right) + g_{ij}(z, \frac{dz}{dt})\left(\frac{d^2 z^j}{dt^2} + 2G^j(z, \frac{dz}{dt})\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \eta^i}\right) - \frac{\partial L}{\partial z^i}, \quad i = \overline{1, n},$$

for some complex Finsler functions L.

ii. Projective metrizability $\rightsquigarrow$

- Projectively related $(2,0)$-homogeneous complex sprays
$\rightsquigarrow$ S and $\tilde{S}$ are projectively related iff there is a $(1,0)$-homogeneous function $\mathcal{P}(z,\eta)$ on $T'M$ such that $\tilde{G}^k = G^k + \mathcal{P}(z,\eta)\eta^k$.
- Projective Finsler metrizable homogeneous complex spray i.e. when it is projectively related to a weakly Kähler Finsler metrizable spray.

► Another direction $\rightsquigarrow$ whether it is possible for two projectively related complex Finsler metrics to have the same $h\bar{h}$-curvature tensor $\rightsquigarrow$ to find answer for non Kähler-Berwald spaces.

3. **Navigation problems on Hermitian manifolds** (M, h)

- W-Zermelo deformation when W is a gradient vector field (i.e. $W = h^{\bar{m}i}\frac{\partial \omega}{\partial \bar{z}^m}\frac{\partial}{\partial z^j}$, $\omega : M \to \mathbb{R}$ is a smooth real valued function on M).
- A Hermitian approach for Matsumoto's slope-of-a-mountain problem.

② Future research directions related to navigation problems

4. Geometric properties of the $(\eta, \tilde{\eta})$-slope metric $\rightsquigarrow$ flag curvature, Ricci curvature, projective flatness, Einstein conditions, Douglas conditions, etc.

- $\mathbf{G}^T$ is a gradient vector field $\rightsquigarrow$ the differential 1-form β, defined in (39), is closed $\Rightarrow$ each $(\eta, \tilde{\eta})$-slope metric becomes a candidate to be a Douglas metric on an n-dimensional manifold with $n \geq 3$ [X. M. Wang, B. Li, Acta Math. Sinica 2017].
- To study the geodesics of the Finsler spaces with $(\eta, \tilde{\eta})$-slope metrics when the gravitational wind $\mathbf{G}^T$ is an infinitesimal homothety, i.e. $\mathcal{L}_{\mathbf{G}^T} h = \sigma h$, where σ is a constant $\rightsquigarrow$ to see if it is possible to obtain a classification of $(\eta, \tilde{\eta})$-slope metrics of constant flag curvature.
- Under assumption $\mathcal{L}_{\mathbf{G}^T} h = \sigma(x) h$ (i.e. $\mathbf{G}^T$ is conformal to h), to study if there exist $(\eta, \tilde{\eta})$-slope metrics that are projectively flat or projectively related to α.

5. Non-uniform slippery slope

- One or both traction coefficients depend on the position, $\eta = \eta(x)$ or/and $\tilde{\eta} = \tilde{\eta}(x)$ $\rightsquigarrow$ more extensive resultant Finsler metrics $\tilde{F}_{\eta\tilde{\eta}} = \alpha\tilde{\phi}_{\eta\tilde{\eta}}$ because $\tilde{\phi}_{\eta\tilde{\eta}}$ depends in addition on a third variable or on two more variables.
- A varying self-speed $||u||_h$ of a craft on a slippery slope (M, h), i.e. $||u||_h = f(x)$, where f is a smooth function on M and $f(x) \in (0, 1]$, for any $x \in M$.

② Future research directions related to navigation problems

6. Matsumoto's slope-of-a-mountain problem with wind

Consider the navigation data (F, W) on the Finsler manifold (M, F), where:

- $F = \frac{\alpha^2}{\alpha - \bar{g}\beta}$ is the Matsumoto metric with $||G^T||_h < \frac{1}{2}$;
- W is a vector field (the wind in the sense of Zermelo's navigation).

$\rightsquigarrow$ a Finsler metric $\tilde{F}$ implicitly given by

$$\tilde{F}\left(\sqrt{||y||_h^2 - 2h(y, W)\tilde{F} + ||W||_h^2\tilde{F}^2} + h(y, \mathbf{G}^T) - h(W, \mathbf{G}^T)\tilde{F}\right) = ||y||_h^2 - 2h(y, W)\tilde{F} + ||W||_h^2\tilde{F}^2,$$

under the constrains $h(W, G^T) < ||W||_h(1 - ||W||_h)$ and $||G^T||_h < \frac{1}{2}$ which assure the strong convexity of the indicatrix $I_{\tilde{F}}$.

7. Slippery slope of a mountain in the presence of a wind

$\rightsquigarrow$ the time-minimal paths on (M, h) in the presence of an active wind $\mathbf{G}_\eta$ and an arbitrary wind W are the geodesics of the Finsler metric $\mathcal{F}_\eta$ which satisfies

$$\mathcal{F}_\eta\left\{(||W||_h^2 + \Omega_1)\mathcal{F}_\eta^2 - 2[h(y, W) + (1-\eta)h(y, \mathbf{G}^T)]\mathcal{F}_\eta + ||y||_h^2\right\}^{1/2}$$
$$= (||W||_h^2 + \Omega_2)\mathcal{F}_\eta^2 - [2h(y, W) + (2-\eta)h(y, \mathbf{G}^T)]\mathcal{F}_\eta + ||y||_h^2,$$

under the restrictions $||W||_h^2 + \Omega_1 < \sqrt{||W||_h^2 + \Omega_2}$ and $||\mathbf{G}^T||_h < \tilde{b}_0$,
with $\Omega_1 = (1-\eta)[2h(W, \mathbf{G}^T) + (1-\eta)||\mathbf{G}^T||_h^2]$, $\Omega_2 = (2-\eta)h(W, \mathbf{G}^T) + (1-\eta)||\mathbf{G}^T||_h^2$ and either $\tilde{b}_0 = 1$ if $\eta \in \left[0, \frac{1}{2}\right]$ or $\tilde{b}_0 = \frac{1}{2\eta}$ if $\eta \in \left(\frac{1}{2}, 1\right]$.

③ A brief list of the candidate's background

Q1 and Q2 research publications:

⇝ according to the ranking AIS lists, editions 2020-2024 (JCR 2019 - JCR 2023)

- **9 papers published in Q1 journals**:

  1 ⇝ The Journal of Geometric Analysis (Jul. 2025)

  2 ⇝ Nonlinear Analysis-Theory Methods and Applications (Feb. 2023, Oct. 2023)

  2 ⇝ Nonlinear Analysis-Real World Applications (Oct. 2012, Feb. 2025)

  1 ⇝ Journal of the Franklin Institute-Eng. and Applied Mathematics (Jan. 2021)

  1 ⇝ Journal of Optimization Theory and Applications (Apr. 2021)

  1 ⇝ Annual Reviews in Control (2020)

  1 ⇝ The Journal of Navigation (Jan. 2021)

- **10 papers published in Q2 journals**:

  3 ⇝ Journal of Geometry and Physics (Feb. 2012, Apr. 2013, Aug. 2016)

  3 ⇝ Results in Mathematics (Sep. 2016, Dec. 2017, Aug. 2020)

  2 ⇝ Differential Geometry and its Applications (Dec. 2012, Oct. 2017)

  1 ⇝ Periodica Mathematica Hungarica (Mar. 2023)

  1 ⇝ Acta Mathematica Scientia (Jul. 2014)

③ A brief list of the candidate's background $\rightsquigarrow$ Further perspectives

$\rightsquigarrow$ W. r. t. the new Romanian minimal standards for habilitation [OMEC 3019 (February 11, 2025)] which will be applied starting with October 1, 2026, I have the parameters:

S_1=9,1668 (≥ 4), **S_2=8,5578** $(\geq 2,5)$, **C_1=35** (≥ 20), **C_2=27** (≥ 10), **N_{recent}=16** ≥ 2, (counted in February 2025).

Financial support:

- grant 2013/09/N/ST10/02537 financed by the Polish National Science Center, Jagiellonian University in Krakow (2016-2017)
- postdoctoral program POSDRU/89/1.5/S/59323 financed by the European Social Fund of the Romanian Goverment (2010-2013)
- Transilvania University of Brașov grant (2012)
- grant CNCSIS A 424/2006.

Didactic activities:

- 2015-present $\rightsquigarrow$ advisor for 37 bachelor or master theses in Differential geometry and Linear algebra.

$\Downarrow$

Overall candidate's perspectives:

- to extend and enhance the research significantly in the aforementioned directions,
- to explore new avenues that may contribute to conferring a higher academic position.

Thank you very much!