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NEW CONTRIBUTIONS TO FRACTAL INTERPOLATION THEORY

ABSTRACT

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Contents

Introduction

Considerations on fractal interpolation theory

Interpolation serves the purpose of recovering a function when only specific points from its graph are accessible. Traditional interpolation methods, that use polynomial, rational, exponential, trigonometric or spline functions, produce interpolation functions that are piece-wise differentiable. However, these functions are not appropriate for the vast majority of real-world situations where the data exhibit irregularities and lack smoothness in their behavior.

Fractal interpolation differs from other conventional types of interpolation methods because the continuous interpolation function obtained is not necessarily differentiable at any point. Consequently, fractal interpolation provides interpolants that are closer to natural world phenomena, thus proving to be a more versatile tool for fitting real-world data. Moreover, it provides a broad spectrum of interpolants, ranging from those that are nowhere differentiable to infinitely differentiable ones. The concept of fractal interpolation functions was originally introduced by M. Barnsley (see [5] and [6]) and it has been the subject of extensive research ever since.

Fractal interpolation is a distinctive interpolation method that consists of constructing a continuous function that passes through all of the points of a provided system of data, with its graph being the attractor of an iterated function system. More precisely, Barnsley proved that given a finite real subset A and a function $f : A \to \mathbb{R}$, there exists a continuous function $F : [\min A, \max A] \to \mathbb{R}$ such that:

a) $F_{|A} = f$;

b) there exists an iterated function system whose attractor is the graph of F.

The function F is called a fractal interpolation function (for short FIF) corresponding to the set of data $\{(a, f(a)) : a \in A\}$.

The initial results proved by Barnsley regarding FIFs were studied in depth leading to numerous generalizations and novel research directions. Among these directions of research, we mention:

a) multivariable fractal interpolation functions which are obtained via higher dimensional or recurrent iterated function systems (see [8]);

b) hidden variable fractal interpolation functions involving the projection of the attractors of vector-valued iterated function systems to some lower dimensional spaces (see [7], [13], [18], [19], [45], [91] and [94]);

c) Hermite or spline fractal interpolation functions (see [58] and [94]);

d) bilinear fractal interpolants which are based on bilinear functions (see [11]);

e) fractal splines which combine fractal functions and splines (see [9] and [47]);

f) fractal interpolation surfaces (see [14], [15], [24], [30], [44], [46], [76], [79], [92], [97] and [102]);

g) generalizations of Barnsley's fractal interpolation technique for a countable set of data (see [31], [70], [82], [83], [84], [85], [86] and [93]).

For comprehensive and useful expository accounts of fractal interpolation one can consult [46] and [63].

Motivation for choosing the theme of the thesis

Fractal interpolation is based on a constructive method, via an iterative procedure, as opposed to other classical interpolation methods (such as linear, polynomial, Hermite or spline based methods) that rely on a descriptive method. Moreover, fractal interpolation allows working both with smooth, as well as non-smooth approximation, thus being more suitable for real-world applications.

FIFs have various applications in significant areas of research. Among them, we mention:

- image compression (see [10] and [25]);

- image upscaling (see [69]);

- video image compression (see [1]);
- satellite image data reconstruction (see [20]);
- image encryption (see [100]);
- theory of Schauder bases (see [61] and [64]);
- signal processing (see [60], [62] and [101]);
- fingerprint shape reconstruction (see [3]);
- tumor perfusion reconstruction (see [17]);
- quantification of cognitive brain processes (see [59]);
- financial analysis (see [39]);
- stock price index prediction (see [96]);
- seismic data reconstruction (see [41]);
- rock fracture surfaces (see [98]);
- prediction of river dissolved oxygen in complex watershed (see [42]);
- prediction of wind speed (see [99] and [103]);
- study of epidemics (see [2] and [67]);
- refining the quality of data in the preprocessing step of Machine Learning prediction algorithms (see [73]).

The versatility of fractal interpolation is underlined by the variety of applications that use it. Moreover, the theory of fractal interpolation has an increasing interest among the research community since there are a multitude of engaging different research directions (for example, the directions of research a)-g) from the previous section).

The motivation for choosing the theme of the current thesis relies on the significant interest that fractal interpolation has already proven to have among the research community and on the growing amount of fields where fractal interpolation applications arise.

Structure of the thesis

In this thesis, we present new contributions to fractal interpolation theory, organized into six chapters. We begin by establishing the fundamental notation and terminology in the first chapter, followed by a detailed study of the Read-Bajraktarevic operator in the second chapter. The third chapter explores the concept of countable fractal interpolation, while the fourth chapter introduces a novel framework for fractal interpolation. In the fifth chapter, we introduce a new type of iterated function system, and in the final chapter, we apply fractal interpolation to real-world scenarios, including an analysis of Covid-19 spread.

The first chapter is dedicated to collecting the main notation and terminology that will be used throughout the thesis, as well as some key concepts that are essential in understanding the contents of the thesis. We introduce fundamental notions related to generalized contractions, iterated function systems, the shift space and the canonical projection.

Chapter two is devoted to the study of the Read-Bajraktarevic operator, a fundamental concept within the field of fractal interpolation theory. The main results contained in this chapter are part of the paper **"Scale-free fractal interpolation"**, *"Fractal Fract."* **6** (2022) (see [66]), which is published in collaboration with Maria Navascués and Vasileios Drakopoulos. We present some properties of the Read-Bajraktarevic operator in the case of finite data sets. Moreover, we study the conditions under which the Read-Bajraktarevic operator produces smooth interpolation functions for a given set of data.

In the third chapter, there are presented results related to countable fractal interpolation that are published in the article **"A countable fractal interpolation scheme involving Rakotch contractions"**, *"Results Math."* **76** (2021) (see [68]). This chapter is dedicated to FIFs for countable systems of data that present themselves as the attractor of a countable iterated function system which has more general constitutive functions. More precisely, the constitutive functions are Rakotch contractions, thus, they are not necessarily Banach contractions. We prove that for a countable set of data, there exists a continuous interpolation function whose graph is the attractor of a countable iterated function system composed of Rakotch contractions. In the final part of the chapter, we give some examples of particular cases of countable iterated function systems involving Rakotch contractions.

In the subsequent chapter, which constitutes the fourth chapter of this thesis, we present the contents of the article **"A fractal interpolation**

scheme for a possible sizeable set of data", *"J. Fractal Geom."* **9** (2022) (see [55]) that is a joint work with Radu Miculescu and Alexandru Mihail. Within this chapter, we introduce a more comprehensive framework for fractal interpolation. Specifically, given $a, b \in \mathbb{R}$, $a < b$, and $A \subseteq \mathbb{R}$ such that $\{a, b\} \in A = \overline{A} \subseteq [a, b]$ and the interior of A is empty, we prove that for every continuous function $f : A \to \mathbb{R}$, there exist a continuous function $g^* : [a, b] \to \mathbb{R}$ and a possibly infinite iterated function system whose attractor is the graph of g^* such that $g^*|_A = f$. In other words, our results prove the existence of a FIF corresponding to the set of data $\{(x, f(x)) : x \in A\}$. Notably, our scheme allows A to be uncountable as is the case of the Cantor ternary set. As a result, our findings make a substantial contribution to the field of FIFs.

The fifth chapter introduces a new type of iterated function system, more specifically interpolation type iterated function system, and presents the content of the article **"Interpolation type iterated function systems"**, *"J. Math. Anal. Appl."* **519** (2023) (see [56]) that is published in collaboration with Radu Miculescu and Alexandru Mihail. We collect several properties of interpolation type iterated function systems, demonstrating that such a system has attractor and admits canonical projection. As a by-product of our results, we establish a fixed point result.

The final chapter is dedicated to applications of fractal interpolation and contains results from the articles **"An analysis of COVID-19 spread based on fractal interpolation and fractal dimension"**, *"Chaos Solitons Fractals"* **139** (2020) (see [67]) and **"A concretization of an approximation method for non-affine fractal interpolation functions"**, *"Mathematics"* **9** (2021) (see [16]). In the first part of the chapter, we present an application of fractal interpolation in retrieving the missing data registered in the first months (first half of 2020) of the Covid-19 pandemic. Moreover, we employ the box-counting dimension as a measure to evaluate the complexity of the spread of Covid-19. In the second part of the chapter, we present two algorithms, one deterministic and one probabilistic, that allow visualizations of approximations of the FIF obtained via the scheme presented in chapter three. Thus, the second part of the chapter presents an application of an approximation technique for FIFs that involve Rakotch contractions.

In summary, the current thesis presents a comprehensive exploration of fractal interpolation, covering fundamental concepts, novel frameworks and real-world applications. Thus, our contribution enhances both the theoretical knowledge and practical significance of the field.

Original results contained in the thesis

The main original results contained in the present thesis are the following:

A. *A new countable fractal interpolation scheme involving Rakotch contractions*

The main novelty brought in the research field related to FIFs is using countable iterated function systems composed of Rakotch contractions. We prove that there exists a FIF that interpolates countable data, whose graph is the attractor of such a countable iterated function system.

B. *A fractal interpolation scheme for possibly sizeable data*

We introduce new results that extend Barnsley's fractal interpolation technique. More precisely, the main original result that we introduce states that for $a, b \in \mathbb{R}$, $a < b$ and $A \subseteq \mathbb{R}$ such that $\{a, b\} \in A = \overline{A} \subseteq [a, b]$ and $\overset{\circ}{A} = \emptyset$, given a continuous function $f : A \to \mathbb{R}$, there exists a FIF that interpolates the data $\{(a, f(a)) : a \in A\}$. We emphasize the fact that our results allow the set A to be uncountable (as is the case of the Cantor ternary set), which is a significant improvement brought to the theory of FIFs.

C. *The concept of interpolation type iterated function system*

We introduce a novel concept, that of interpolation type iterated function system. The new notion emerges from the theory of FIFs. Within the framework of this newly introduced interpolation type iterated function system, we establish two significant findings: we prove that such a system has attractor and that it admits canonical projection.

Moreover, we provide a correlated fixed-point result that is obtained as a by-product of our main results.

D. *Smooth interpolation functions generated by Read-Bajraktarevic type operators*

We prove that Read-Bajraktarevic type operators could provide smooth interpolation functions for certain systems of data.

E. *Applications of fractal interpolation*

We present an application of fractal interpolation in the study of epidemics. More precisely, we use fractal interpolation to retrieve missing data related to the first months of the Covid-19 pandemic and we utilize the box-counting dimension as a measure to evaluate the complexity of the spread of Covid-19.

Also, we present a deterministic algorithm and a probabilistic one, that allow visualizations of approximations of FIF.

Dissemination of the results

The original results mentioned in the preceding section (**A**, **B**, **C**, **D** and **E**) were disseminated in the mathematical community both in the form of papers published in significant international journals and as oral communications at conferences, workshops or symposiums, as follows:

A. In the framework of the "7th International Workshop on Nonlinear Analysis, Fixed Point Theory & Applications", XGEN conference, on Wednesday, 19 May 2021, I presented a talk entitled *"A countable fractal interpolation scheme involving Rakotch contractions"*.

During the 4th edition of the "International Conference on Mathematics and Computer Science" (MACOS) held between 15 -17 September 2022 in Braşov, Romania, on Thursday, 16 September 2022, I delivered the presentation entitled *"A countable fractal interpolation scheme involving Rakotch contractions"*.

I published the paper:

C.M. Păcurar, *"A countable fractal interpolation scheme involving Rakotch contractions"* in "Results in Mathematics" 76 (2021), 161.

and in collaboration with A. Băicoianu and M. Păun the paper *"A Concretization of an Approximation Method for Non-Affine Fractal Interpolation Functions"* in "Mathematics" 9 (2021), 767.

B. In the framework of the "44th Summer Symposium in Real Analysis" held between 20 and 24 June 2022 in Paris & Orsay, on Friday, 24 June 2022, I delivered the talk entitled *"New Contributions to Fractal Interpolation Theory"*.

I published, in collaboration with R. Miculescu and A. Mihail, the paper *"A fractal interpolation scheme for a possible sizeable set of data"* in "Journal of Fractal Geometry" 9 (2022), 337–355.

C. In the framework of the "14th International Conference on Fixed Point Theory and its Applications" held between 11 and 14 July 2023 in Braşov, Romania, on Thursday, 13 July 2023, I delivered the talk entitled *"Interpolation type iterated function systems"*.

I published, in collaboration with R. Miculescu and A. Mihail, the paper *"Interpolation type iterated function systems"* in "Journal of Mathematical Analysis and Applications" 519 (2023), 519, 126747.

D. In the framework of the "International Conference on Approximation Theory and its Applications", held between 12-14 September 2022 in Sibiu, Romania, I delivered an oral presentation with the title *"On some operators appearing in fractal interpolation theory"*.

I published, in collaboration with M. Navascués and V. Drakopoulos, the paper *"Scale-Free Fractal Interpolation"* in "Fractal and Fractional" 6 (2022), 602.

E. I published, in collaboration with B. Necula, the paper *"An analysis of COVID-19 spread based on fractal interpolation and fractal dimension"* in "Chaos Solitons and Fractals" 139 (2020).

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1. Preliminaries

The current chapter collects the notation and terminology that are essential for reading and understanding the current thesis, as well as the key notions that are of utmost importance for the results contained in the following chapters.

1.1 Notation and terminology

In the sequel $\mathbb{N}$ designates the set $\{1, 2, ...\}$.

For a function $f : A \to B$, by G_f we mean the graph of f, i.e. the set $\{(a, f(a)) : a \in A\}$.

For a function $f : X \to X$ and $n \in \mathbb{N}$, we denote the n-times composition of f with itself by $f^{[n]}$.

For a set of indexes I, a family of functions $(f_i)_{i\in I} : X \to X$, and $k \in \mathbb{N}$ fixed, we denote by $f_{i_1 i_2 \dots i_k}$ the composition of the functions f_{i_k}, i.e. $f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}$, where $(i_j)_{j\in\{1,\dots,k\}} \subseteq I$.

For a metric space (X, d) and $A \subseteq X$, we shall use the following notation:

- $\sup\limits_{x,y\in A} d(x, y) \coloneqq diam(A)$;
- $\{A \subseteq X : A \neq \emptyset \text{ and } A \text{ is bounded}\} \coloneqq P_b(X)$;
- $\{A \subseteq X : A \neq \emptyset \text{ and } A \text{ is closed}\} \coloneqq P_{cl}(X)$;
- $P_b(X) \cap P_{cl}(X) \coloneqq P_{b,cl}(X)$;
- $\{A \subseteq X : A \neq \emptyset \text{ and } A \text{ is compact}\} \coloneqq P_{cp}(X)$.

Additionally, for $A, B \in P_b(X)$ and $x \in X$, we will also use the following notation:

- $\inf\limits_{a\in A} d(x, a) \coloneqq d(x, A)$
- $\sup\limits_{a\in A} d(a, B) \coloneqq d(A, B)$.

Definition 1.1. Let (X, d) be a metric space. For a function $f : X \to X$, we define the Lipschitz constant of f as

$$\sup_{x,y\in X,\, x\neq y} \frac{d(f(x),f(y))}{d(x,y)} := lip(f) \in [0,+\infty].$$

If $lip(f) < +\infty$, then the function f is called Lipschitz.

For $a, b \in \mathbb{R}$, $a < b$, (Y, ρ) a metric space and $\alpha, \beta \in Y$, we consider the sets

$$\{f : [a,b] \to Y : f \text{ is continuous}\} := C_Y([a,b])$$

and

$$\{f : [a,b] \to Y : f \text{ is continuous, } f(a) = \alpha \text{ and } f(b) = \beta\} := C_Y^{\alpha,\beta}([a,b]).$$

For $(Y, \rho) = (\mathbb{R}, |\cdot|)$ we shall use the notation

$$C_{\mathbb{R}}([a,b]) = C([a,b])$$

and

$$C_{\mathbb{R}}^{\alpha,\beta}([a,b]) = C^{\alpha,\beta}([a,b]).$$

All of the above spaces of functions, endowed with the uniform metric d_u (i.e. $d_u(g,h) = \sup\limits_{x\in[a,b]} \rho(g(x),h(x))$ for every $g, h \in C_Y([a,b])$) are complete.

For $k \in \{0\} \cup \mathbb{N}$, the set

$$\{f : [a,b] \to \mathbb{R} \,:\, f \text{ is k times differentiable and } f^{(k)} \in C([a,b])\}$$

is denoted by $C^k([a,b])$, where by $f^{(0)}$ we mean f.

Endowed with the norm $\|f\|_k = \max\limits_{p\in\{0,1,\ldots,k\}} \|f^{(p)}\|_\infty = \max\limits_{p\in\{0,1,\ldots,k\}} \sup\limits_{x\in[a,b]} |f^{(p)}(x)|$, $C^k([a,b])$ is complete.

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $k \leq n$. We consider the partial or incomplete Bell polynomial (see [12]) given by

$$B_{n,k}(x_1; \ldots; x_{n-k+1}) =$$

$$= \sum_{\substack{j_1+j_2+\cdots+j_{n-k+1}=k \\ j_1+2j_2+\cdots+(n-k+1)j_{n-k+1}=n}} \frac{n!}{j_1! \ldots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}.$$

For $f, g \in C^k([a,b])$, according to the Faà di Bruno formula (see [23] and [28]), we have

$$(f \circ g)^{(p)}(x) = \sum_{i=1}^{p} f^{(i)}(g(x)) B_{p,i}(g^{(1)}(x); g^{(2)}(x); \ldots; g^{(p-i+1)}(x)), \qquad (1.1)$$

for all $x \in [a,b]$ and $p \in \{1, 2, \ldots, k\}$.

1.2 Generalized contractions

Definition 1.2. Let (X, d) be a metric space. A map $f : X \to X$ is called contraction (or Banach contraction) if there exists $C \in [0, 1)$ such that

$$d(f(x), f(y)) \leq Cd(x, y),$$

for all x, $y \in X$.

Definition 1.3. (see [34], [48], [72] and [74]) Let $\varphi : [0, \infty) \to [0, \infty)$ and (X, d) a metric space. A map $f : X \to X$ is called:

i) φ-contraction if

$$d(f(x), f(y)) \leq \varphi(d(x, y)),$$

for all x, $y \in X$.

ii) Rakotch contraction if it is a φ-contraction, with the function φ such that the function $\alpha : (0, \infty) \to (0, \infty)$, given by $\alpha(t) = \frac{\varphi(t)}{t}$ for every $t > 0$ is non-increasing and $\alpha(t) < 1$ for every $t > 0$.

iii) Browder contraction if it is a φ-contraction, where the function φ is non-decreasing, $\varphi(t) < t$ for every $t > 0$ and φ is right-continuous.

iv) Matkowski contraction if it is a φ-contraction, where the function φ is non-decreasing and $\lim\limits_{n\to\infty} \varphi^{[n]}(t) = 0$ for all $t > 0$.

Remark 1.1 (see the diagram on page 144 from [34])**.**

i) Each Banach contraction is a Rakotch contraction (for a function φ given by $\varphi(t) = \alpha t$ for every $t \geq 0$, where $\alpha \in [0, 1)$).

ii) Each Rakotch contraction is a Browder contraction.

iii) Each Browder contraction is a Matkowski contraction.

iv) The above two statements ensure that each Rakotch contraction is a Matkowski contraction.

Definition 1.4. Given a metric space (X, d), an operator $f : X \to X$ is called a Picard operator if it has a unique fixed point $x_* \in X$ and

$$\lim_{n\to\infty} f^{[n]}(x) = x_*,$$

for every $x \in X$.

Theorem 1.1 (see [4])**.** *Given a complete metric space (X, d), if $f : X \to X$ is a Banach contraction, then f is a Picard operator.*

Theorem 1.2 (see Theorem 1.2 from [48])**.** *Given a complete metric space (X, d), if $f : X \to X$ is a Matkowski contraction, then f is a Picard operator.*

1.3 Iterated function systems

The concept of iterated function system is a notion due to Hutchinson (see [33]). His theory created a rigorous theoretical framework for obtaining fractals. Given this, many research studies addressed the subject.

Definition 1.5. Let (X, d) be a metric space. The function

$$h : P_{b,cl}(X) \times P_{b,cl}(X) \to [0, \infty),$$

defined as

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

for every $A, B \in P_{b,cl}(X)$, which is a metric, is called the Hausdorff-Pompeiu metric on X.

Definition 1.6. Let (X, d) be a complete metric space, I a finite set and the family of continuous functions $(f_i)_{i\in I}$ where $f_i : X \to X$. The pair $((X, d), (f_i)_{i\in I})$ is called an iterated function system. In the sequel, we shall call such a system, for short, IFS.

We refer to such an IFS as $\mathcal{S} = ((X, d), (f_i)_{i\in I})$.

The fractal operator associated to the IFS $\mathcal{S}$ is the function $F_{\mathcal{S}} : P_{cp}(X) \to P_{cp}(X)$, defined as

$$F_{\mathcal{S}}(K) = \bigcup_{i\in I} f_i(K),$$

for every $K \in P_{cp}(X)$.

If the fractal operator $F_{\mathcal{S}}$ is Picard, then we say that the IFS $\mathcal{S}$ has attractor and the fixed point of $F_{\mathcal{S}}$ is called the attractor of the IFS $\mathcal{S}$. We denote the attractor by $A_{\mathcal{S}} \in P_{cp}(X)$.

1.3.1 Countable iterated function systems

The concept of IFS can be generalized for a countable family of constitutive functions. Thus arose the concept of countable iterated function system (see [29], [43], [81] and [87]).

Definition 1.7. Let (X, d) be a compact metric space and $f_n : X \to X$ be continuous functions for every $n \in \mathbb{N}$. The pair $((X, d), (f_n)_{n\in\mathbb{N}})$ is called a countable iterated function system. In the sequel, we shall call such a system, for short, CIFS.

We refer to such a CIFS as $\mathcal{S}_C = ((X,d),(f_n)_{n\in\mathbb{N}})$.

The fractal operator associated to the CIFS $\mathcal{S}_C$ is the function $F_{\mathcal{S}_C} : P_{cp}(X) \to P_{cp}(X)$, defined as

$$F_{\mathcal{S}_C}(K) = \overline{\bigcup_{n\in\mathbb{N}} f_n(K)},$$

for every $K \in P_{cp}(X)$.

If the fractal operator $F_{\mathcal{S}_C}$ is Picard, then we say that the CIFS $\mathcal{S}_C$ has attractor and the fixed point of $F_{\mathcal{S}_C}$ is called the attractor of the CIFS $\mathcal{S}_C$. We denote the attractor by $A_{\mathcal{S}_C} \in P_{cp}(X)$.

Theorem 1.3 (see Theorem 4.6 from [86] and Theorem 3.9 from [88]). *If the constitutive functions f_n of the CIFS $\mathcal{S}_C = ((X,d),(f_n)_{n\in\mathbb{N}})$ are Matkowski contractions for the functions $\varphi_n : [0,\infty) \to [0,\infty)$ for every $n \in \mathbb{N}$ such that $\sup\limits_{n\in\mathbb{N}} \varphi_n(t) < t$ for every $t \geq 0$, then $\mathcal{S}_C$ has attractor.*

1.3.2 Possibly infinite iterated function systems

A more general concept is that of possibly infinite iterated function systems. Such systems encompass both the case of finite and countable number of constitutive and, moreover, accommodate the case of uncountable infinite number of constitutive functions.

Definition 1.8. Let (X,d) be a complete metric space and a family of functions $(f_i)_{i\in I}$ with the following properties:

- $f_i : X \to X$ are Banach contractions such that $\sup\limits_{i\in I} lip(f_i) < 1$,

- the family of functions $(f_i)_{i\in I}$ is bounded, i.e. $\bigcup\limits_{i\in I} f_i(A) \in P_b(X)$ for every $A \in P_b(X)$.

The pair $((X,d),(f_i)_{i\in I})$ is called a possibly infinite iterated function system. In the sequel, we shall call such a system, for short, PIIFS.

We refer to such a PIIFS as $\mathcal{S}_I = ((X,d),(f_i)_{i\in I})$.

The fractal operator associated to the PIIFS $\mathcal{S}_I$ is the function $F_{\mathcal{S}_I} : P_{b,cl}(X) \to P_{b,cl}(X)$, given by

$$F_{\mathcal{S}_I}(B) = \overline{\bigcup_{i\in I} f_i(B)},$$

for all $B \in P_{b,cl}(X)$.

Theorem 1.4 (see Theorem 4.1 from [57]). *For each $\mathcal{S}_I = ((X,d),(f_i)_{i\in I})$, there exists a unique $A_{\mathcal{S}_I} \in P_{b,cl}(X)$, called the attractor of $\mathcal{S}_I$, such that*

$$F_{\mathcal{S}_I}(A_{\mathcal{S}_I}) = A_{\mathcal{S}_I}.$$

In addition, we have

$$\lim_{n\to\infty} h(F_{\mathcal{S}_I}^{[n]}(B), A_{\mathcal{S}_I}) = 0,$$

for every $B \in P_{b,cl}(X)$.

1.4 The shift space and the canonical projection

Let I be a non-empty set and $n \in \mathbb{N}$. Throughout the thesis, we employ the following notation:
- $I^{\mathbb{N}} := \Lambda(I)$
- $I^{\{1,2,\ldots,n\}} := \Lambda_n(I)$.

$\Lambda(I)$ is the set of infinite words with letters from the alphabet I and a standard element ω of $\Lambda(I)$ has the form $\omega = \omega_1\omega_2...\omega_n\omega_{n+1}...$.

$\Lambda_n(I)$ is the set of words of length n with letters from the alphabet I and a standard element ω of $\Lambda_n(I)$ has the form $\omega = \omega_1\omega_2...\omega_n$.

$\Lambda(I)$ endowed with the distance described by

$$d_\Lambda(\omega,\theta) = \begin{cases} 0, & \text{if } \omega = \theta \\ \frac{1}{2^{\min\{k\in\mathbb{N}\,:\,\omega_k\neq\theta_k\}}}, & \text{if } \omega \neq \theta \end{cases},$$

where $\omega = \omega_1\omega_2\omega_3...\omega_n\omega_{n+1}...$ and $\theta = \theta_1\theta_2\theta_3...\theta_n\theta_{n+1}...$, becomes a metric space.

For $m \in \mathbb{N}$ and $\omega = \omega_1\omega_2...\omega_n\omega_{n+1}... \in \Lambda(I)$, we shall use the following notation in the sequel

$$\omega_1\omega_2...\omega_m := [\omega]_m.$$

Let us note that if I is finite, then the metric space $(\Lambda(I), d_\Lambda)$ is compact. If I is infinite, then the metric space $(\Lambda(I), d_\Lambda)$ is complete.

For $i \in I$, we can consider the function $\tau_i : \Lambda(I) \to \Lambda(I)$ given by

$$\tau_i(\omega) = i\omega_1\omega_2...\omega_n\omega_{n+1}..., \tag{1.2}$$

for every $\omega = \omega_1\omega_2...\omega_n\omega_{n+1}... \in \Lambda(I)$.

Given $f_i : X \to X$, $i \in I$, and $\omega = \omega_1\omega_2\ldots\omega_n \in \Lambda_n(I)$, the following notation will be used in the sequel:

$$f_{\omega_1\omega_2\ldots\omega_n} := f_\omega.$$

In the particular case that I has just one element, let us call it i, we have

$$f_{\underbrace{i\ldots i}_{n \text{ times}}} = f_i^{[n]}$$

for every $n \in \mathbb{N}$.

The canonical projection associated to an IFS is an onto function that takes the shift space $\Lambda(I)$ to the attractor of the IFS considered. The canonical projection allows alternative characterizations of the attractor of an IFS and it is a useful tool in the study of fractals.

Definition 1.9. For the IFS $\mathcal{S} = ((X, d), (f_i)_{i \in I})$, having attractor, we say that it admits canonical projection if:

i) For every $\omega = \omega_1\omega_2\ldots\omega_n\omega_{n+1}\ldots \in \Lambda(I)$, $\lim\limits_{n\to\infty} f_{\omega_1\omega_2\ldots\omega_n}(x)$ - denoted by $\pi(\omega)$ - exists and does not depend on $x \in X$.

ii) $\pi(\omega) \in A_{\mathcal{S}}$ for every $\omega \in \Lambda(I)$.

iii) The function $\pi : \Lambda(I) \to A_{\mathcal{S}}$ has the following properties:

   a) it is continuous;

   b) it is onto;

   c) $\pi \circ \tau_i = f_i \circ \pi$ for every $i \in I$.

Remark 1.2. *The function π from Definition 1.9, iii) is called the canonical projection from $\Lambda(I)$ to $A_{\mathcal{S}}$.*

2. The Read-Bajraktarevic operator

In this chapter, we study Read-Bajraktarevic type operators that are essential in the fractal interpolation theory. The final part of the chapter is dedicated to the study of Read-Bajraktarevic type operators that provide smooth interpolation functions.

2.1 Interpolation functions

2.1.1 Finite systems of data

Let $I \subseteq \mathbb{R}$ a real compact and let us consider the finite system of points

$$\Gamma = \{(x_n, y_n) \in I \times \mathbb{R} \,:\, n \in \{0, \ldots, N\}\},$$

where $N \in \mathbb{N}$.

If $x_{n-1} < x_n$ for every $n \in \{1, 2, \ldots, N\}$, the system of points Γ is called a system of data. In this case, let us consider $I = [x_0, x_N]$.

Definition 2.1. An interpolation function corresponding to the system of data Γ is a continuous function $f : I \to \mathbb{R}$ such that $f(x_n) = y_n$, for each $n \in \{0, 1, \ldots, N\}$.

Let the family of affine homeomorphisms $k_n : I \to [x_{n-1}, x_n]$ be such that

$$k_n(x_0) = x_{n-1} \quad \text{and} \quad k_n(x_N) = x_n,$$

for every $n \in \{1, 2, \ldots, N\}$ and there exists $\zeta_n \in [0, 1)$ such that

$$|k_n(x) - k_n(x')| \leq \zeta_n |x - x'|,$$

for every $x, x' \in I$ and every $n \in \{1, 2, \ldots, N\}$.

Let the family of continuous functions $F_n : I \times \mathbb{R} \to \mathbb{R}$ be such that

$$F_n(x_0, y_0) = y_{n-1} \quad \text{and} \quad F_n(x_N, y_N) = y_n,$$

for every $n \in \{1, 2, \ldots, N\}$.

For $n \in \{1, 2, \ldots, N\}$, we define $K_n : I \times \mathbb{R} \to I \times \mathbb{R}$ as

$$K_n(x, y) = (k_n(x), F_n(x, y)),$$

for every $x \in I$ and $y \in \mathbb{R}$.

The notion of a system of data can be generalized to that of a countable system of data as follows.

2.1.2 Countable systems of data

Let (Y, d) be a compact metric space and let us consider the countable system of points

$$\Delta = \{(x_n, y_n) \in \mathbb{R} \times Y \;:\; n \in \{0\} \cup \mathbb{N}\}. \tag{2.1}$$

The system of points defined in relation (2.1) is called a countable system of data if the sequence $(x_n)_{n \in \{0\} \cup \mathbb{N}}$ is strictly increasing and bounded, and the sequence $(y_n)_{n \in \{0\} \cup \mathbb{N}}$ is convergent.

We establish the following notations

$$x_0 = a, \quad \lim_{n \to \infty} x_n = b, \quad y_0 = m, \quad \lim_{n \to \infty} y_n = M.$$

Definition 2.2. In the context described above, a continuous function $f : [a, b] \to Y$ such that $f(x_n) = y_n$, for each $n \in \{0\} \cup \mathbb{N}$ is called an interpolation function corresponding to the countable system of data Δ.

Let $\Delta = \{(x_n, y_n) \in [a, b] \times Y \;:\; n \in \{0\} \cup \mathbb{N}\}$ be a countable system of data.

For each $n \in \mathbb{N}$, let $l_n : [a, b] \to [x_{n-1}, x_n]$ be a homeomorphism for which there exists $L_n \in [0, 1)$ such that

i) $|l_n(x) - l_n(x')| \leq L_n |x - x'|$ for every x, $x' \in [a, b]$;

ii) $l_n(a) = x_{n-1}$ and $l_n(b) = x_n$;

iii) $\sup_{n \in \mathbb{N}} L_n < 1$.

For each $n \in \mathbb{N}$, let $W_n : [a, b] \times Y \to Y$ be a continuous function such that

j) $W_n(a, m) = y_{n-1}$ and $W_n(b, M) = y_n$;

jj) $\lim\limits_{n\to\infty} diam(Im\, W_n) = 0$.

For $n \in \mathbb{N}$, we define $f_n : [a, b] \times Y \to [a, b] \times Y$ as

$$f_n(x, y) = (l_n(x), W_n(x, y)),$$

for every $x \in [a, b]$ and $y \in Y$.

2.2 Read-Bajraktarevic type operators

2.2.1 The Read-Bajraktarevic operator for the finite case

Let us consider the context from section 2.1.1 and the space $C^{y_0,y_N}(I)$ endowed with the uniform metric d_u. In the sequel, we will use the notation:

$$C^{y_0,y_N}(I) = \mathcal{C}.$$

Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be the Read-Bajraktarevic operator defined as

$$\mathcal{T}(f)(x) = F_n(k_n^{-1}(x), f(k_n^{-1}(x))),$$

for every $f \in \mathcal{C}$ and every $x \in [x_{n-1}, x_n]$.

The operator $\mathcal{T}$ is well defined and

$$\mathcal{T}(f)(x_n) = y_n,$$

for every $f \in \mathcal{C}$ and every $n \in \{0, 1, \ldots, N\}$.

The following theorem proves that under certain conditions, the Read-Bajraktarevic operator $\mathcal{T}$ has a unique fixed point that is an interpolant for the data Γ.

Theorem 2.1. *In the previous context, let F_n be Matkowski contractions in the second variable, i.e. there exist the non-decreasing functions $\varphi_n : [0, \infty) \to [0, \infty)$ such that $\lim\limits_{m\to\infty} \varphi_n^{[m]}(t) = 0$ for all $t > 0$ and*

$$|F_n((x, y)) - F_n((x, y'))| \le \varphi_n(|y - y'|)$$

for all $x \in I$ and $y, y' \in \mathbb{R}$. If the map $\varphi : [0, \infty) \to [0, \infty)$ defined as $\varphi(t) = \sup\limits_{n\in\{1,2,\ldots,N\}} \varphi_n(t)$ is such that $\lim\limits_{m\to\infty} \varphi^{[m]}(t) = 0$ for every $t > 0$, then the operator $\mathcal{T}$ is a Matkowski contraction, and consequently, it has a unique fixed point $f_* \in \mathcal{C}$ that is an interpolant for the data Γ.

2.2.2 The Read-Bajraktarevic operator for the countable case

Let us consider the context from section 2.1.2. We consider the space $C_Y^{m,M}([a,b])$ endowed with the uniform metric d_u. We will denote $C_Y^{m,M}([a,b])$ by $\mathfrak{C}$.

Let Δ be a countable system of data noted as in (2.1).

For $f \in \mathfrak{C}$, we consider the function $\mathbf{T}_f : [a,b] \to Y$ given as follows:

$$\mathbf{T}_f(x) = \begin{cases} W_n(l_n^{-1}(x), f(l_n^{-1}(x))), & \text{if } x \in [x_{n-1}, x_n], \quad n \in \mathbb{N} \\ M, & \text{if } x = b. \end{cases}$$

Since $\mathbf{T}_f$ is well defined and $\mathbf{T}_f \in \mathfrak{C}$, the operator $T : \mathfrak{C} \to \mathfrak{C}$, defined as

$$T(f) = \mathbf{T}_f$$

for every $f \in \mathfrak{C}$ is well defined.

2.2.3 Smooth interpolation functions

Even though fractal interpolation is significantly important for modeling irregular and non-smooth data, that require functions which are not differentiable at any point, it also encompasses smooth functions. More precisely, this entails functions that are k-times continuously differentiable that seamlessly traverse a given set of data points. The current section presents results that are part of [66].

Let $k \in \mathbb{N}$ be arbitrary, but fixed. Let $a, b \in \mathbb{R}$, $a < b$ and the real interval $I = [a,b]$. For $N \in \mathbb{N}$, $N > 1$, let us consider $x_i \in [a,b]$ for every $i \in \{0, 1, \ldots, N\}$ such that $x_0 = a$, $x_N = b$ and $x_{i-1} < x_i$ for every $i \in \{1, 2, \ldots, N\}$. We will denote by I_i the interval $[x_{i-1}, x_i]$ for every $i \in \{1, 2, \ldots, N\}$.

We consider the finite system of data

$$\Delta' = \{(x_i, y_{i,p}) \in I \times \mathbb{R} : i \in \{0, 1, \ldots, N\}, p \in \{0, 1, \ldots, k\}\}.$$

Let us consider $l_i : I \to I_i$ defined as

$$l_i(x) = \frac{x_i - x_{i-1}}{b - a} x + \frac{b x_{i-1} - a x_i}{b - a} = a_i x + b_i,$$

for every $x \in I$.

Let us consider $S_i \in C^k(I)$ and $R_i \in C^k(\mathbb{R})$, where $i \in \{1, 2, \ldots, N\}$. For $f \in C^k(I)$ such that $f^{(p)}(x_i) = y_{i,p}$ for every $i \in \{0, 1, \ldots, N\}$ and

$p \in \{0, 1, \dots, k\}$, let the family of continuous mappings $W_i : I \times \mathbb{R} \to \mathbb{R}$, defined as

$$W_i(x, y) = f(l_i(x)) + R_i(y) - S_i(x),$$

for every $(x, y) \in I \times \mathbb{R}$ and every $i \in \{1, 2, \dots, N\}$.

Suppose that for every $i \in \{1, 2, \dots, N\}$ and $p \in \{1, 2, \dots, k\}$ the following conditions are satisfied:

$$S_i^{(p)}(a) = \sum_{j=1}^{p} R_i^{(j)}(y_{0,0}) B_{p,j}(y_{0,1}; y_{0,2} \dots ; y_{0,p-j+1}), \tag{2.2}$$

$$S_i^{(p)}(b) = \sum_{j=1}^{p} R_i^{(j)}(y_{N,0}) B_{p,j}(y_{N,1}; y_{N,2} \dots ; y_{N,p-j+1}), \tag{2.3}$$

$$S_i(a) = R_i(y_{0,0}) \quad \text{and} \quad S_i(b) = R_i(y_{N,0}),$$

where $B_{p,j}$ are the Bell polynomials defined in section 1.1.

Let us consider

$$\mathcal{A}^k(I) = \{g \in C^k(I) \,:\, g^{(p)}(a) = y_{0,p} \text{ and } g^{(p)}(b) = y_{N,p} \text{ for every } p \in \{0, 1, \dots, k\}\},$$

and endow it with the norm $\|\cdot\|_k$ (defined as in section 1.1).

Let the Read-Bajraktarevic type operator $\mathcal{D} : \mathcal{A}^k(I) \to \mathcal{A}^k(I)$ be defined as usual by

$$\begin{aligned}\mathcal{D}(g)(x) &= W_i(l_i^{-1}(x), g(l_i^{-1}(x))),\\ &= f(x) + R_i(g(l_i^{-1}(x))) - S_i(l_i^{-1}(x)),\end{aligned}$$

for every $x \in I_i$ and every $g \in \mathcal{A}^k(I)$.

Proposition 2.1. *The operator $\mathcal{D}$ is well defined.*

Theorem 2.2 (see Theorem 5. from [66])**.** *In framework mentioned above, there exists $f^R \in C^k(I)$ satisfying the following functional equations:*

$$\begin{aligned}&(f^R)^{(p)}(x) = f^{(p)}(x) +\\ &+ a_i^{-p} \sum_{j=1}^{p} R_i^{(j)}(f^R(l_i^{-1}(x))) B_{p,j}((f^R)^{(1)}(l_i^{-1}(x)); \dots ; (f^R)^{(p-j+1)}(l_i^{-1}(x))) -\\ &- a_i^{-p} S_i^{(p)}(l_i^{-1}(x)) = f^{(p)}(x) + a_i^{-p}(R_i \circ f^R)^{(p)}(l_i^{-1}(x)) - a_i^{-p} S_i^{(p)}(l_i^{-1}(x)),\end{aligned} \tag{2.4}$$

for every $i \in \{1, 2, \dots, N\}$, $p \in \{1, \dots, N\}$, $x \in I_i$, and

$$f^R(x) = f(x) + (R_i \circ f^R)(l_i^{-1}(x)) - S_i(l_i^{-1}(x)), \tag{2.5}$$

for every $x \in I_i$. The function f^R interpolates the data Δ'.

3. A countable fractal interpolation scheme involving Rakotch contractions

A direction of interest regarding FIFs centers around the utilization of fixed point results that extend beyond Banach's well-established theorem (see Theorem 1.1) to guarantee the existence of the FIF. Historically, after Barnsley's pioneering work, many extensions of fractal interpolation relied predominantly on the Banach fixed point theorem to establish the existence of FIFs. However, recent developments have opened up new possibilities by employing alternative fixed point theorems. In this context, notable contributions have emerged, from which we mention that of S. Ri who used Rakotch contractions to obtain new results (see [75]), J. Kim et al. who resorted to Geraghty contractions (see [38]) and S. Ri and V. Drakopoulos who extended the results to surfaces (see [77]). These advancements highlight the diversification of mathematical tools and techniques used in the pursuit of FIF existence, offering new avenues for exploration in this area of research.

Another direction related to FIFs involves considering countable sets of points instead of finite ones. In this respect, countable fractal interpolation has been introduced by N. Secelean (see [82]) based on CIFSs (see [29], [81], [87]). Secelean proved the existence of the FIF for a set of data $\Delta = \{(x_n, y_n) \in I \times \mathbb{R} : n \in \{0\} \cup \mathbb{N}\}$ where $(x_n)_{n\in\{0\}\cup\mathbb{N}}$ is a strictly increasing bounded sequence with $b = \lim\limits_{n\to\infty} x_n$ such that $I = [x_0, b]$ and $(y_n)_{n\in\{0\}\cup\mathbb{N}}$ is a convergent sequence (see [82]). A. Gowrisankar and R. Uthayakumar extended these findings to encompass data characterized by $(x_n)_{n\in\{0\}\cup\mathbb{N}}$ being a monotone bounded sequence and $(y_n)_{n\in\{0\}\cup\mathbb{N}}$ a bounded sequence (see [31]). This direction of research has seen further development in subsequent papers, including [83], [84] and [93], highlighting the growing interest and advancements in the study of countable fractal interpolation.

In the current chapter, we combine the distinct two lines of research initiated by Secelean and Ri. In this way, we introduce a new fractal interpolation scheme for countable systems of data and CIFSs composed of Rakotch contractions. As a consequence, the findings presented in this chapter represent an expansion of the results previously established in [75] and [82], which offer a broader perspective on fractal interpolation. The contents of the current chapter are based on the results from **"A countable fractal interpolation scheme involving Rakotch contractions"**, published in *"Results in Mathematics"* by **C. Păcurar** (see [68]).

3.1 Fractal interpolation functions involving Rakotch contractions

Let (Y, d) be a compact metric space and let us consider the families of functions $(l_n)_{n\in\mathbb{N}}$, $(W_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ and the family of numbers $(L_n)_{n\in\mathbb{N}}$ defined as in section 2.1.2.

The operator T is the Read-Bajraktarevich operator for the countable case, introduced in section 2.2.2, for the countable system of data (2.1).

The first result proves that the operator T is a Matkowski contraction if the functions W_n are Matkowski contractions in the second argument. Thus, since $(\mathfrak{C}, d_u)$ is complete, the next theorem proves that the operator T has a unique fixed point provided that the functions W_n are Matkowski contractions in the second argument.

Theorem 3.1 (see Theorem 3 from [68]). *Let $\Delta = \{(x_n, y_n) \in \mathbb{R} \times Y : n \in \{0\} \cup \mathbb{N}\}$ be a countable system of data. If the functions W_n are Matkowski contractions with respect to the second argument, i.e. there exists a non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$ and*

$$d(W_n((x, y)), W_n((x, y'))) \leq \varphi(d(y, y')) \tag{3.1}$$

for all $x \in [a, b]$ and $y, y' \in Y$, then T is a Matkowski contraction.

In the next result we will prove that if the functions W_n are Lipschitz with respect to the first variable and Rakotch contractions in the second variable, then the functions f_n are Rakotch contractions with respect to a metric that is equivalent to the initial metric on $[a, b] \times Y$. The functions f_n are the constitutive functions of the CIFS that will provide a FIF.

Theorem 3.2 (see Theorem 4 from [68]). *Let $\Delta = \{(x_n, y_n) \in \mathbb{R} \times Y : n \in \{0\} \cup \mathbb{N}\}$ be a countable system of data such that W_n are Lipschitz with respect to the first variable and Rakotch contractions in the second variable, i.e. there exist $L > 0$, and a non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that the function $\alpha : (0, \infty) \to (0, \infty)$, given by $\alpha(t) = \frac{\varphi(t)}{t}$, for every $t > 0$ is non-increasing and $\alpha(t) < 1$ for every $t > 0$, such that*

$$d(W_n((x, y)), W_n((x', y'))) \leq L|x - x'| + \varphi(d(y, y')),$$

for all (x, y), $(x', y') \in [a, b] \times Y$, $n \in \mathbb{N}$.

Then, f_n are Rakotch contractions with respect to the metric d_θ described by

$$d_\theta((x, y), (x', y')) \coloneqq |x - x'| + \theta d(y, y'),$$

for all (x, y), $(x', y') \in [a, b] \times Y$, where $\theta = \frac{1 - \sup\limits_{n \in \mathbb{N}} L_n}{2(L+1)} \in (0, 1)$.

Remark 3.1 (see Remark 5 from [68]). *Let $\Delta = \{(x_n, y_n) \in \mathbb{R} \times Y : n \in \{0\} \cup \mathbb{N}\}$ be a countable system of data. If the functions f_n are Rakotch contractions with respect to the metric d_θ (in particular, if the conditions stated in Theorem 3.2 are satisfied), then the CIFS $\mathcal{S}_\mathcal{C} = (([a, b] \times Y, d_\theta), (f_n)_{n \in \mathbb{N}})$ has attractor, so there exists a unique $A_{\mathcal{S}_\mathcal{C}} \in P_{cp}([a, b] \times Y)$ such that*

$$F_{\mathcal{S}_C}(A_{\mathcal{S}_C}) = A_{\mathcal{S}_C}.$$

The next theorem is the main result of the current chapter as it proves that there exists an interpolation function for a countable system of data, such that its graph is the attractor of a CIFS. More precisely, the following result proves the existence of a FIF in the aforementioned context.

Theorem 3.3 (see Theorem 5 from [68]). *Let $\Delta = \{(x_n, y_n) \in \mathbb{R} \times Y : n \in \{0\} \cup \mathbb{N}\}$ be a countable system of data such that the functions W_n satisfy the hypothesis from Theorem 3.2. Then there exists an interpolation function f_* corresponding to Δ such that its graph is the attractor of the CIFS $\mathcal{S}_C = (([a, b] \times Y, d_\theta), (f_n)_{n \in \mathbb{N}})$.*

Theorem 3.4 (see Theorem 6 from [68]). *In the context of Theorem 3.3, we have*

$$\lim_{n \to \infty} G_{T^{[n]}(f_0)} = G_{f_*},$$

for every $f_0 \in \mathfrak{C}$.

3.2 Particular cases of countable iterated function systems involving Rakotch contractions

The current section provides some particular cases of CIFSs involving Banach and Rakotch contractions.

In the context of section 2.1.2, we can choose

$$l_n(x) = \frac{x_n - x_{n-1}}{b-a}x + \frac{bx_{n-1} - ax_n}{b-a},$$

for every $x \in [a, b]$ and for every $n \in \mathbb{N}$.

Let us present two ways to choose the functions W_n, provided that $Y \subseteq [0, \infty)$:

A.

$$W_n(x, y) = \left(\frac{y_n - y_{n-1}}{b-a} - d_n\frac{M-m}{b-a}\right)x + d_n y + \frac{by_{n-1} - ay_n}{b-a} - d_n\frac{bm - aM}{b-a} =$$

$$= c_n x + d_n y + g_n,$$

where $d_n \in [0, 1)$ such that $\lim\limits_{n\to\infty} d_n = 0$, for every $n \in \mathbb{N}$.

W_n are Banach contractions with respect to the second variable, which implies that they are Rakotch contractions on the second variable for the function φ given by $\varphi(t) = c \cdot t$ for every $t \geq 0$.

In this case, the functions f_n are as follows:

$$f_n(x, y) = \left(\frac{x_n - x_{n-1}}{b-a}x + \frac{bx_{n-1} - ax_n}{b-a},\right.$$
$$\left.\left(\frac{y_n - y_{n-1}}{b-a} - d_n\frac{M-m}{b-a}\right)x + d_n y + \frac{by_{n-1} - ay_n}{b-a} - d_n\frac{bm - aM}{b-a}\right),$$

for every $n \in \mathbb{N}$ and every $x \in [a, b]$, $y \in Y$.

B.

$$W_n(x, y) = c_n x + \frac{y}{1+ny} + g_n,$$

where

$$c_n = \frac{y_n - y_{n-1}}{b-a} - \frac{1}{b-a}\left(\frac{M}{1+nM} - \frac{m}{1+nm}\right)$$

and
$$g_n = y_{n-1} - a\frac{y_n - y_{n-1}}{b-a} + \frac{a}{b-a}\frac{M}{1+nM} - \frac{b}{b-a}\frac{m}{1+nm},$$
for every $n \in \mathbb{N}$.

W_n are Rakotch contraction with respect to the second variable for every $n \in \mathbb{N}$.

In this case, the functions f_n can be chosen as follows:

$$\begin{aligned} f_n(x,y) = \Bigg(& \frac{x_n - x_{n-1}}{b-a}x + \frac{bx_{n-1} - ax_n}{b-a}, \\ & \left(\frac{y_n - y_{n-1}}{b-a} - \frac{1}{b-a}\left(\frac{M}{1+nM} - \frac{m}{1+nm}\right)\right)x + \frac{y}{1+ny} + \\ & + y_{n-1} - a\frac{y_n - y_{n-1}}{b-a} + \frac{a}{b-a}\frac{M}{1+nM} - \frac{b}{b-a}\frac{m}{1+nm}\Bigg), \end{aligned}$$

for every $n \in \mathbb{N}$ and every $x \in [a,b]$, $y \in Y$.

4. A fractal interpolation scheme for a possibly sizeable set of data

In this chapter, we extend Barnsley's fractal interpolation method. We consider $a, b \in \mathbb{R}$, $a < b$, $A \subseteq \mathbb{R}$ such that

$$\{a, b\} \subseteq A = \overline{A} \subseteq [a, b]$$

and additionally, we require that the interior of A, denoted as $\overset{\circ}{A}$, is empty. Our central result asserts that for every continuous function $f : A \to \mathbb{R}$ there exist a continuous function $g^* : [a, b] \to \mathbb{R}$ and a PIIFS whose attractor is the graph of g^* and such that $g^*_{|A} = f$. In other words, our main result ensures the existence of a FIF corresponding to the set of data $\{(a, f(a)) : a \in A\}$.

If A is finite we obtain Barnsley's well-established interpolation scheme, as described in [5]. On the other hand, when A takes the form $A = \{x_n : n \in \mathbb{N}\} \cup \{b\}$, where $(x_n)_{n\in\mathbb{N}}$ is strictly increasing, $x_1 = a$, $\lim\limits_{n\to\infty} x_n = b$ and $x_n \in [a, b]$ for every $n \in \mathbb{N}$, we obtain the interpolation scheme presented by Secelean in [82]. We emphasize the fact that our interpolation scheme accommodates situations where A can be uncountable, as it is the case of the Cantor ternary set.

The main tool to address the challenges concerning the step between countable and uncountable data is the theorem concerning the structure of open subsets of $\mathbb{R}$. This theorem provides a sequence $(I_n)_{n\in\mathbb{N}}$ of open disjoint intervals having the property that

$$[a, b] \smallsetminus A = \bigcup_{n\in\mathbb{N}} I_n.$$

Subsequently, via this sequence, we consider an operator

$$T : C^{f(a),f(b)}([a, b]) \to C^{f(a),f(b)}([a, b]).$$

The most challenging aspect that arises within the context of uncountable data is proving that T is well defined, that is ensuring that $T(g) \in C^{f(a),f(b)}([a,b])$ for each $g \in C^{f(a),f(b)}([a,b])$ (see Proposition 4.1), because we have to overcome some technical difficulties.

The results contained in the current chapter, which are based on the paper **"A fractal interpolation scheme for a possible sizeable set of data"** that I published in collaboration with R. Miculescu and A. Mihail in *"Journal of Fractal Geometry"* (see [55]), are more general than the results previously existing in the literature.

4.1 Some technical results

Let us consider $a, b \in \mathbb{R}$, $a < b$ and $A \subseteq \mathbb{R}$ having the following properties:

i) $\{a, b\} \subseteq A = \overline{A} \subseteq [a, b]$;

ii) $\overset{\circ}{A} = \emptyset$.

Then there exists a sequence $(I_n)_{n\in\mathbb{N}}$ of open disjoint intervals such that

$$[a, b] \setminus A = \bigcup_{n\in\mathbb{N}} I_n.$$

where

$$I_n = (\alpha_n, \beta_n),$$

for every $n \in \mathbb{N}$.

We study separately the accumulation points of $A \cap (x, \infty)$, respectively $A \cap (-\infty, x)$, and the points that are not accumulation points.

Remark 4.1 (see Remark 3.2. from [55]). *a) If $x \in A$ is not an accumulation point of $A \cap (x, \infty)$, then there exists $n \in \mathbb{N}$ such that $x = \alpha_n$.*

b) In a similar way, if x is not an accumulation point of $A \cap (-\infty, x)$, then there exists $n \in \mathbb{N}$ such that $x = \beta_n$.

Remark 4.2 (see Remark 3.3. from [55]). *a) If $x \in A$ is an accumulation point of $A \cap (x, \infty)$, then for every sequence $(x_k)_{k\in\mathbb{N}} \subseteq (x, b) \setminus A$ having the property that $\lim\limits_{k\to\infty} x_k = x$, there exists a sequence $((\alpha_{n_k}, \beta_{n_k}))_{k\in\mathbb{N}}$ of elements from the family $\{(\alpha_n, \beta_n) : n \in \mathbb{N}\}$ such that:*

i) $x < \alpha_{n_k} < \beta_{n_k}$, for every $k \in \mathbb{N}$;

ii) $x_k \in (\alpha_{n_k}, \beta_{n_k})$, for every $k \in \mathbb{N}$;

iii) the set $\{x_k : k \in \mathbb{N}\} \cap (\alpha_{n_k}, \beta_{n_k})$ *is finite for every* $k \in \mathbb{N}$*;*

iv) $\lim\limits_{k\to\infty} \alpha_{n_k} = \lim\limits_{k\to\infty} \beta_{n_k} = x$*.*

b) If $x \in A$ *is an accumulation point of* $A \cap (-\infty, x)$*, then for every sequence* $(x_k)_{k\in\mathbb{N}} \subseteq (a, x) \setminus A$ *having the property that* $\lim\limits_{k\to\infty} x_k = x$*, there exists a sequence* $((\alpha_{n_k}, \beta_{n_k}))_{k\in\mathbb{N}}$ *of elements from the family* $\{(\alpha_n, \beta_n) : n \in \mathbb{N}\}$ *such that:*

i) $\alpha_{n_k} < \beta_{n_k} < x$*, for every* $k \in \mathbb{N}$*;*

ii) $x_k \in (\alpha_{n_k}, \beta_{n_k})$*, for every* $k \in \mathbb{N}$*;*

iii) the set $\{x_k : k \in \mathbb{N}\} \cap (\alpha_{n_k}, \beta_{n_k})$ *is finite for every* $k \in \mathbb{N}$*;*

iv) $\lim\limits_{k\to\infty} \alpha_{n_k} = \lim\limits_{k\to\infty} \beta_{n_k} = x$*.*

4.2 Fractal interpolation functions associated to possibly sizeable sets of data

We consider the functions $l_n : [a, b] \to [\alpha_n, \beta_n]$ given by

$$l_n(x) = \frac{\beta_n - \alpha_n}{b - a} x + \frac{\alpha_n b - \beta_n a}{b - a} = a_n x + b_n,$$

for every $x \in [a, b]$ and every $n \in \mathbb{N}$.

Remark 4.3 (see Remark 3.4. from [55]). *For the functions* l_n*, the following hold:*

a) $l_n(a) = \alpha_n$ *and* $l_n(b) = \beta_n$ *for every* $n \in \mathbb{N}$*.*

b) $l_n^{-1} : [\alpha_n, \beta_n] \to [a, b]$ *is given by*

$$l_n^{-1}(x) = \frac{b - a}{\beta_n - \alpha_n} x + \frac{\beta_n a - \alpha_n b}{\beta_n - \alpha_n},$$

for every $x \in [\alpha_n, \beta_n]$ *and every* $n \in \mathbb{N}$*.*

c) $l_n^{-1}(\alpha_n) = a$ *and* $l_n^{-1}(\beta_n) = b$ *for every* $n \in \mathbb{N}$*.*

For a continuous function $f : A \to \mathbb{R}$, we can consider the functions $g_n : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g_n(x, y) =$$

$$= \left(\frac{f(\beta_n) - f(\alpha_n)}{b-a} - d_n \frac{f(b) - f(a)}{b-a} \right) x + d_n y + \frac{bf(\alpha_n) - af(\beta_n)}{b-a} - d_n \frac{bf(a) - af(b)}{b-a} =$$

$$= c_n x + d_n y + e_n,$$

for every $(x, y) \in \mathbb{R}^2$ and every $n \in \mathbb{N}$, where $(d_n)_{n\in\mathbb{N}} \subseteq [0, 1)$ is such that

$$\lim_{n\to\infty} d_n = 0.$$

Remark 4.4 (see Remark 3.5. from [55])**.** *For the functions g_n, the following conditions are satisfied:*

$$g_n(a, f(a)) = f(\alpha_n) \text{ and } g_n(b, f(b)) = f(\beta_n),$$

for every $n \in \mathbb{N}$.

Let us consider the space $C^{f(a),f(b)}([a, b])$ endowed with the uniform metric d_u. We will denote $C^{f(a),f(b)}([a, b])$ by $\mathbf{C}$.

For $g \in \mathbf{C}$, let us consider the function $T_g : [a, b] \to \mathbb{R}$, given by

$$T_g(x) = \begin{cases} f(x), & x \in A \\ c_n l_n^{-1}(x) + d_n g(l_n^{-1}(x)) + e_n, & \text{if } x \in (\alpha_n, \beta_n) \end{cases}.$$

Remark 4.5 (see Remark 3.6. from [55])**.** *a) We have*

$$T_g(\alpha_n) = c_n l_n^{-1}(\alpha_n) + d_n g(l_n^{-1}(\alpha_n)) + e_n,$$

for every $g \in \mathbf{C}$ and every $n \in \mathbb{N}$.

b) We have

$$T_g(\beta_n) = c_n l_n^{-1}(\beta_n) + d_n g(l_n^{-1}(\beta_n)) + e_n,$$

for every $g \in \mathbf{C}$ and every $n \in \mathbb{N}$.

Proposition 4.1 (see Proposition 3.7. from [55])**.** *In the above framework, we have*

$$T_g \in \mathbf{C},$$

for every $g \in \mathbf{C}$.

Proposition 4.1 allows us to define the operator $T : \mathbf{C} \to \mathbf{C}$ given by

$$T(g) = T_g,$$

for every $g \in \mathbf{C}$.

Proposition 4.2 (see Proposition 3.8. from [55]). *In the above mentioned framework, we have*

$$d_u(T(g_1), T(g_2)) \leq (\sup_{n\in\mathbb{N}} d_n)\, d_u(g_1, g_2),$$

for all $g_1, g_2 \in \mathbf{C}$, so T is a Banach contraction with respect to the uniform metric d_u.

As $(\mathbf{C}, d_u)$ is a complete metric space, using Theorem 1.1, Proposition 4.2 ensures that there exists a unique $g^* \in \mathbf{C}$ such that

$$T(g^*) = g^*.$$

Remark 4.6 (see Remark 3.9. from [55]). *We have*

$$g^*_{|A} = f.$$

We are going to prove that there exists a PIIFS whose attractor is the graph of g^*.

Let us consider the functions $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f_n(x, y) = (a_n x + b_n, c_n x + d_n y + e_n),$$

for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$.

Remark 4.7 (see Remark 3.10. from [55]).

a) We have $\sup_{n\in\mathbb{N}} a_n < 1$.

b) The sequence $(c_n)_{n\in\mathbb{N}}$ is bounded.

Remark 4.7 allows us to consider

$$\theta \in \left(0, \frac{1 - \sup_{n\in\mathbb{N}} a_n}{C}\right),$$

$C > 0$ being such that $|c_n| \leq C$, for every $n \in \mathbb{N}$ and the metric ρ, on $\mathbb{R}^2$, given by

$$\rho((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + \theta\, |y_1 - y_2|,$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

Proposition 4.3 (see Proposition 3.11. from [55]). *In the context described above, the functions f_n are Banach contractions with respect to the metric ρ.*

Remark 4.8 (see Remark 3.12. from [55]). *a) The metric space $(\mathbb{R}^2, \rho)$ is a complete metric space.*

b) We have $\sup\limits_{n\in\mathbb{N}} lip(f_n) \leq \max\{\sup\limits_{n\in\mathbb{N}} a_n + \theta C, \sup\limits_{n\in\mathbb{N}} d_n\} < 1$.

c) The family $(f_n)_{n\in\mathbb{N}}$ is bounded since the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$ are bounded.

In view of Remark 4.8, we can consider the PIIFS (see section 1.3.2.) $\mathcal{S}_I = ((\mathbb{R}^2, \rho), (f_n)_{n\in\mathbb{N}})$.

Let us also consider $G_{g^*} = \{(x, g^*(x)) \,:\, x \in [a, b]\} := G$.

Theorem 4.1 (see Theorem 3.13. from [55]). *In the above mentioned context, we have*

$$G = A_{\mathcal{S}_I}.$$

We will present some examples of sets A that satisfy the conditions mentioned above:

A. If A is finite we obtain the classic Barnsley's interpolation scheme (see [5]).

B. If $A = \{x_n \,:\, n \in \mathbb{N}\} \cup \{b\}$, where $(x_n)_{n\in\mathbb{N}}$ is strictly increasing, $x_1 = a$ and $\lim\limits_{n\to\infty} x_n = b$ we obtain the interpolation scheme presented in [82].

C. We can choose $A = \{x_n \,:\, n \in \mathbb{N}\} \cup \{y_n \,:\, n \in \mathbb{N}\} \cup \{a, b\}$, where $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$, $x_n \in [a, \frac{a+b}{2}]$ and $y_n \in [\frac{a+b}{2}, b]$ for every $n \in \mathbb{N}$.

D. We can choose A to be the Cantor ternary set. This is a significant example since A is not countable. Therefore, our scheme is a genuine generalization of the one presented by Secelean in [82].

The final part of the current chapter presents a result which shows that for every $g \in \mathbf{C}$, if n is big enough, the graph of the FIF can be approximated, as close as we want, by the graph of $T^{[n]}(g)$.

Theorem 4.2 (see Theorem 3.14. from [55]). *In the context described above, we have $F_{\mathcal{S}_I}(G_g) = G_{T(g)}$, for every $g \in \mathbf{C}$.*

Theorem 4.3 (see Theorem 3.15. from [55]). *In the context mentioned above, we have $\lim\limits_{n\to\infty} h(G_{T^{[n]}(g)}, G) = 0$, for every $g \in \mathbf{C}$.*

5. Interpolation type iterated function systems

The current chapter introduces a novel concept of IFS, more precisely, "interpolation type iterated function system". The concept is based on the fundamental characteristics inherent to the IFSs employed in constructing FIFs. We prove several properties that these systems have. Our main result states that such a system has attractor and that it admits canonical projection. Additionally, we provide a correlated fixed point result, that we obtain as a by-product of our main result.

The contents of the current chapter are based on the paper **"Interpolation type iterated function systems"**, published in *"Journal of Mathematical Analysis and Applications"* in collaboration with Radu Miculescu and Alexandru Mihail (see [56]).

5.1 Auxiliary lemmas

The first section is dedicated to some preliminary lemmas that are essential in proving the main results of the current chapter.

Lemma 5.1 (see Lemma 2.7. from [56])**.** *We consider two metric spaces (X, d) and (Y, ρ) and a non-empty finite set I. We suppose that the collections of self-mappings of Y $\{A_{\omega,n} : \omega \in \Lambda(I),\ n \in \mathbb{N}\}$ and $\{A_{\omega,x,n,k} : \omega \in \Lambda(I),\ x \in X,\ n, k \in \mathbb{N}\}$ are such that:*

i)

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N},\omega\in\Lambda(I),x\in K_1,y\in Y} \rho(A_{\omega,x,n,k}(y), A_{\omega,n}(y)) = 0,$$

for every $K_1 \in P_{cp}(X)$.

ii) There exists a function $b : \Lambda(I) \to Y$ such that for every $K_2 \in P_{cp}(Y)$ we have

$$\lim_{n\to\infty} \sup_{\omega\in\Lambda(I), y\in K_2} \rho((A_{\omega,1} \circ \ldots \circ A_{\omega,n})(y), b(\omega)) = 0.$$

iii) There exists $C \in [0,1)$ such that

$$\sup_{n,k\in\mathbb{N},\omega\in\Lambda(I),x\in K_1} lip(A_{\omega,x,n,k}) \leq C,$$

for every $K_1 \in P_{cp}(X)$.

Then, we have

$$\lim_{n\to\infty} \sup_{\omega\in\Lambda(I),x\in K_1,y\in K_2} \rho((A_{\omega,x,1,n} \circ A_{\omega,x,2,n-1} \circ \ldots \circ A_{\omega,x,n-1,2} \circ A_{\omega,x,n,1})(y), b(\omega)) = 0,$$

for every $K_1 \in P_{cp}(X)$ and $K_2 \in P_{cp}(Y)$.

5.2 Interpolation type iterated function systems

Within this section we give the definition of interpolation type iterated function system and provide some examples of such systems.

Let us consider the metric spaces (X, d) and (Y, ρ), and

$$d_{\max}((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), \rho(y_1, y_2)\},$$

for every (x_1, y_1), $(x_2, y_2) \in X \times Y$. We consider the metric space $(X \times Y, d_{\max})$.

Definition 5.1 (see Definition 3.1. from [56])**.** An interpolation type iterated function system $\mathcal{S}$ consists of:

- two complete metric spaces (X, d) and (Y, ρ)
- a finite family of functions $(f_i)_{i\in I}$ such that:
  i) For each $i \in I$, there exist the continuous functions $g_i : X \to X$ and $h_i : X \times Y \to Y$ having the property that $f_i : X \times Y \to X \times Y$ is given by
  $$f_i(x, y) = (g_i(x), h_i(x, y)),$$
  for every $(x, y) \in X \times Y$.

ii) For every $\omega \in \Lambda(I)$, there exists $a_\omega \in X$ such that

$$\lim_{n\to\infty} \sup_{x\in K, \omega\in\Lambda(I)} d(g_{[\omega]_n}(x), a_\omega) = 0,$$

for every $K \in P_{cp}(X)$.

iii) There exists $C \in [0,1)$ having the property that

$$\rho(h_i(x,y_1), h_i(x,y_2)) \leq C\rho(y_1,y_2),$$

for every $x \in X$, $y_1, y_2 \in Y$ and $i \in I$.

We refer to such a system as the pair $\mathcal{S} = ((X \times Y, d_{\max}), (f_i)_{i\in I})$. For the sake of brevity, in the sequel, we will refer to interpolation type iterated function system as I-tIFS.

Remark 5.1. *a) An example of an I-tIFS is* $\mathcal{S} = ((\mathbb{R}^2, \|\cdot\|_\infty), (f_i)_{i\in I})$ *where there exist* $a_i, b_i, c_i, d_i, e_i \in \mathbb{R}$, $a_i, d_i \in [0,1)$ *such that*

$$f_i(x,y) = (a_i x + b_i\,, c_i x + d_i y + e_i),$$

for every $(x,y) \in \mathbb{R}^2$ *and* $i \in I$.

b) A more subtle example than the first one could be constructed by taking the functions g_i *Browder contractions that are not Banach contractions. To illustrate this, we present another example of I-tIFS: let* $\mathcal{S} = (([0,1] \times \mathbb{R}, |\cdot|), (f_i)_{i\in\{1,2\}})$, *where there exist* $a_i, b_i, c_i \in \mathbb{R}$, $b_i \in [0,1)$, *such that*

$$f_i(x,y) = (\sin x\,, a_i x + b_i y + c_i),$$

for every $(x,y) \in [0,1] \times \mathbb{R}$ *and* $i \in \{1,2\}$.

c) Let us present two examples of I-tIFSs for which the metric spaces (X,d) *and* (Y,ρ) *are compact.*

The first one is $(([0,1] \times [0,1], \|\cdot\|_\infty), (f_i)_{i\in\{1,2\}})$ *where*

$$f_1(x,y) = \left(\frac{x+1}{2}, \frac{2x+y+1}{4}\right) \quad \textit{and} \quad f_2(x,y) = \left(\frac{x+2}{4}, \frac{x+y+2}{4}\right)$$

for every $x, y \in [0,1]$.

The second one is $(([0,1] \times [0,1], \|\cdot\|_\infty), (f_i)_{i\in\{1,2\}})$ *where*

$$f_1(x,y) = \left(\frac{x+7}{8}, \frac{x+2y+1}{4}\right) \quad \textit{and} \quad f_2(x,y) = \left(\sin x, \frac{3x+2y+1}{6}\right)$$

for all $x, y \in [0,1]$.

5.3 Properties of interpolation type iterated function systems

The current section collects results that show properties of I-tIFSs. These properties are of utmost importance in the proof of the main result, which states that I-tIFSs have attractor.

Let us consider an I-tIFS $\mathcal{S} = ((X \times Y, d_{\max}), (f_i)_{i \in I})$ and two collections of self-mappings of Y $\{A_{\omega,n} : \omega \in \Lambda(I),\, n \in \mathbb{N}\}$ and $\{A_{\omega,x,n,k} : \omega \in \Lambda(I),\, x \in X,\, n, k \in \mathbb{N}\}$ given by

$$A_{\omega,x,n,k}(y) = \begin{cases} h_{\omega_n}(g_{\omega_{n+1}\omega_{n+2}\ldots\omega_{n+k-1}}(x), y), & \text{if } k \geq 2 \\ h_{\omega_n}(x, y), & \text{if } k = 1 \end{cases}$$

and

$$A_{\omega,n}(y) = h_{\omega_n}(a_{\omega_{n+1}\omega_{n+2}\ldots\omega_m\ldots}(x), y)$$

for every $y \in Y$.

Lemma 5.2 (see Lemma 3.5. from [56]). *In the context described above we have*

$$f_{[\omega]_n}(x, y) = (g_{[\omega]_n}(x), (A_{\omega,x,1,n} \circ A_{\omega,x,2,n-1} \circ \ldots \circ A_{\omega,x,n-1,2} \circ A_{\omega,x,n,1})(y)),$$

for every $x \in X$, $y \in Y$ and $n \in \mathbb{N}$, $n \geq 2$.

Lemma 5.3 (see Lemma 3.6. from [56]). *In the previously mentioned framework we have*

$$\lim_{n\to\infty} \sup_{n \in \mathbb{N}, \omega \in \Lambda(I), x \in K_1, y \in K_2} \rho(A_{\omega,x,n,k}(y), A_{\omega,n}(y)) = 0,$$

for every $K_1 \in P_{cp}(X)$ and $K_2 \in P_{cp}(Y)$.

Lemma 5.4 (see Lemma 3.7. from [56]). *In the context described above we have*

$$\sup_{n,k \in \mathbb{N}, \omega \in \Lambda(I), x \in K} lip(A_{\omega,x,n,k}) \leq C,$$

for every $K \in P_{cp}(X)$.

Lemma 5.5 (see Lemma 3.8. from [56]). *In the context described above there exists a function $b : \Lambda(I) \to Y$ such that*

$$\lim_{n\to\infty} (A_{\omega,1} \circ \ldots \circ A_{\omega,n})(y) = b(\omega),$$

for every $y \in Y$.

Moreover,

$$\lim_{n\to\infty} \sup_{\omega\in\Lambda(I), y\in K} \rho((A_{\omega,1} \circ \ldots \circ A_{\omega,n})(y), b(\omega)) = 0,$$

for every $K \in P_{cp}(Y)$.

5.4 Interpolation type iterated function systems have attractor and admit canonical projection

In this section we establish the main result of the current chapter, which states that every I-tIFS has attractor and admits canonical projection.

Proposition 5.1 (see Proposition 4.1. from [56]). *Let $S = ((X \times Y, d_{\max}), (f_i)_{i\in I})$ be an I-tIFS such that (Y, ρ) is compact. Then there exists a function $b : \Lambda(I) \to Y$ such that for every $K \in P_{cp}(X \times Y)$ we have*

$$\lim_{n\to\infty} \sup_{\omega\in\Lambda(I), (x,y)\in K} d_{\max}(f_{[\omega]_n}(x, y), (a_\omega, b(\omega))) = 0.$$

In the context of the above Proposition, using the notation $b(\omega) \coloneqq b_\omega$, we consider the function $\pi : \Lambda(I) \to X \times Y$, described by

$$\pi(\omega) = (a_\omega, b_\omega),$$

for every $\omega \in \Lambda(I)$ and

$$\pi(\Lambda(I)) = \{(a_\omega, b_\omega) \,:\, \omega \in \Lambda(I)\} \coloneqq A_{\mathcal{S}}.$$

Proposition 5.2 (see Proposition 4.2. from [56]). *Let $S = ((X \times Y, d_{\max}), (f_i)_{i\in I})$ be an I-tIFS such that (Y, ρ) is compact. Then:*

a) π is continuous.

b)

$$f_i \circ \pi = \pi \circ \tau_i,$$

for every $i \in I$.

c)

$$F_{\mathcal{S}}(A_{\mathcal{S}}) = A_{\mathcal{S}}.$$

d)

$$A_{\mathcal{S}} \in P_{cp}(X \times Y).$$

e)

$$\lim_{n\to\infty} F_{\mathcal{S}}^{[n]}(K) = A_{\mathcal{S}},$$

for every $K \in P_{cp}(X \times Y)$.

f) $A_{\mathcal{S}}$ is the unique fixed point of $F_{\mathcal{S}}$.

The key findings and implications presented in the preceding Proposition can be effectively summarized in the following central theorem of this chapter:

Theorem 5.1 (see Theorem 4.3. from [56])**.** *Each I-tIFS $S = ((X \times Y, d_{\max}), (f_i)_{i\in I})$ with (Y, ρ) compact has attractor and admits canonical projection.*

5.5 A correlated fixed point result

As an additional result arising from Theorem 5.1, particularly in the case when I has just one element, we obtain the following fixed point result which is related to Theorem 2.2 from [80].

Theorem 5.2 (see Theorem 4.4. from [56])**.** *Let (X, d) be a complete metric space and (Y, ρ) be a compact metric space, $g : X \to X$ and $h : X \times Y \to Y$ continuous such that:*

i) There exists $a \in X$ such that

$$\lim_{n\to\infty} \sup_{x\in K} d(g^{[n]}(x), a) = 0,$$

for every $K \in P_{cp}(X)$.

ii) There exists $C \in [0, 1)$ having the property that

$$\rho(h(x, y_1), h(x, y_2)) \le C\rho(y_1, y_2),$$

for every $x \in X$, $y_1, y_2 \in Y$.

Then $f : X \times Y \to X \times Y$, given by

$$f(x, y) = (g(x), h(x, y)),$$

for every $(x, y) \in X \times Y$, is a Picard operator.

Moreover,

$$\lim_{n\to\infty} \sup_{(x,y)\in K} d_{max}(f^{[n]}(x, y), (a, b)) = 0,$$

for every $K \in P_{cp}(X \times Y)$, where (a, b) is the unique fixed point of f.

6. Applications of fractal interpolation

The current chapter is dedicated to applications of fractal interpolation. The first section is based on the paper **"An analysis of Covid-19 spread based on fractal interpolation and fractal dimension"** published in *"Chaos Solitons Fractals"* (see [67]), in collaboration with B. Necula and the second section is based on the paper **"A Concretization of an Approximation Method for Non-Affine Fractal Interpolation Functions"** published in *"Mathematics"* (see [16]), in collaboration with A. Băicoianu and M. Păun.

In the first section, the Covid-19 pandemic is regarded from a fractal perspective. The epidemiological curves are reconstructed using fractal interpolation upon observing the similarities between the epidemiological curves and some classical fractals (for example, the graph of the Takagi function). Moreover, the box-counting dimension is used to assess the complexity of the evolution of the disease.

In the second section, we study two algorithms, via a probabilistic scheme and a deterministic scheme, that can be used for obtaining FIFs. We study the affine and the non-affine cases for the two algorithms and present the resulting approximation of the FIF that we obtain.

6.1 An analysis of Covid-19 from a fractal perspective

Examining the spread of the Covid-19 pandemic from a fractal point of view could yield a more profound understanding of the disease's intricacies and how it distinguishes itself from historical epidemics. Viewing the spread of the epidemic as a fractal structure holds potential advantages for the medical field, enhancing comprehension of the healthcare crisis induced by

Covid-19, and it can also be a valuable tool for evaluating the development of other epidemics.

In this section, we undertake an examination of the Covid-19 pandemic from a fractal point of view, using fractal interpolation. Additionally, we use the box-counting dimension, also referred to as Minkowski-Bouligand dimension, to asses and quantify the spread of the Covid-19 pandemic across several European countries.

6.1.1 Conclusions

Approaching epidemiological curves as fractals has two main advantages.

Firstly, given the irregularity and challenging to predict nature of the daily increase in the number of cases, considering them as fractals might open a new direction for predicting the evolution of the epidemic. Since the graph is considered a fractal, besides being jagged, it possesses some kind of fractal architecture that is prone to self-similarity. This similarity plays a crucial role in assessing the current status and predicting future changes in epidemiological curves.

On the other hand, treating the epidemiological curve as a fractal and applying fractal interpolation to the available data can serve as a potent tool for data reconstruction. In this respect, we need to emphasize that the number of cases recorded is highly dependent on the number of tests performed by each country on a certain day. Analyzing the data employing fractal interpolation we can cover some pieces of data that might be missing to get a better picture of the epidemic, thereby enhancing our understanding of the epidemic's true extent.

6.2 An approximation of fractal interpolation functions involving Rakotch contractions

The present chapter proposes an implementation of an approximation method for FIFs that involve both affine and non-affine functions. While there have been previous studies addressing the computational aspects of the attractors of IFSs (see [21], [22], [27] and [53]), to the best of our knowledge, there has been no investigation specifically dedicated to non-affine FIFs in this context.

To build the approximation of the FIF, we propose two different algorithms, a probabilistic and a deterministic one.

6.2.1 Two algorithms for computing the fractal interpolation function

We propose two algorithms to obtain the FIF associated to an IFS with affine and non-affine constitutive functions. Algorithm 1 proposes a Probabilistic Scheme, while Algorithm 2 presents a Deterministic Scheme. We list the algorithms below as presented in section 4 from [16].

Algorithm 1 The Probabilistic Scheme.

1: Consider an empty set of points $\mathcal{P} \subseteq \mathbb{R}^2$ and p a significant big signed positive integer.

2: Generate an arbitrary point $(x_c, y_c) \in [0,1] \times [0,1]$.

3: Determine $\mathcal{P} \bigcup \{(x_c, y_c)\}$.

4: Generate a random signed integer $0 < k \leq 100$.

5: Compute $(x_c, y_c) = f_k(x_c, y_c)$.

6: Repeat steps 3, 4 and 5 p times.

7: Sort the elements of the set $\mathcal{P}$ in ascending order with respect to the first component of the elements.

8: Plot the function passing through all the points of the set $\mathcal{P}$.

Algorithm 2 The Deterministic Scheme.

1: Consider k the number of initial points, n the number of functions involved, p the number of steps and define an empty set of points $\mathcal{P} \subseteq \mathbb{R}^2$.

2: Generate randomly a set K_0 of k points in $[0,1] \times [0,1]$.

3: Determine $\mathcal{P} \bigcup K_0$.

4: Compute $\mathcal{P} = f_1(\mathcal{P}) \bigcup f_2(\mathcal{P}) \bigcup \ldots \bigcup f_n(\mathcal{P})$.

5: Repeat step 4 p times.

6: Sort the elements of the set $\mathcal{P}$ in ascending order regarding the first component of the elements.

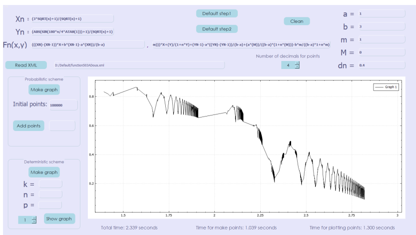
7: Plot the function passing through all the points of the set $\mathcal{P}$.

6.2.2 Results of the approximation algorithms

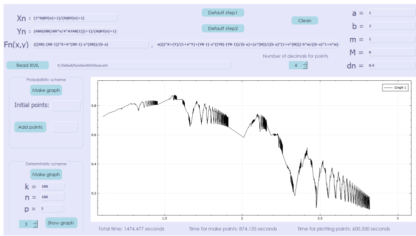
We consider two sequences: $(x_n)_{n\in\{0\}\cup\mathbb{N}}$, given by $x_n = \frac{3\sqrt{n}+1}{\sqrt{n}+1}$, for every $n \in \{0\}\cup\mathbb{N}$, which is positive, increasing and convergent and $(y_n)_{n\in\{0\}\cup\mathbb{N}}$ given by $y_n = \frac{\left|\sin\left(\frac{180\cdot n}{\pi}\right)\right|+1}{\sqrt{n}+1}$, for every $n \in \{0\}\cup\mathbb{N}$, which is convergent. Let $m = \lim\limits_{n\to\infty} x_n = 3$ and $M = \lim\limits_{n\to\infty} y_n = 0$.

Let the sequence of functions $f_n : [x_0, m]\times\mathbb{R} \to [x_0, m]\times\mathbb{R}$, defined as: $f_n(x,y) = \left(\frac{x_n - x_{n-1}}{m-x_0}x + \frac{m x_{n-1} - x_0 x_n}{m-x_0}, \left(\frac{y_n - y_{n-1}}{m-x_0} - - \frac{1}{m-x_0}\left(\frac{M}{1+nM} - \frac{y_0}{1+ny_0}\right)\right)x + + \frac{y}{1+ny} + y_{n-1} - x_0\frac{y_n - y_{n-1}}{m-x_0} + \frac{x_0}{m-x_0}\frac{M}{1+nM} - \frac{m}{m-x_0}\frac{y_0}{1+ny_0}\right)$, for every $n \in \mathbb{N}$, which are non-affine in the second argument.

Using the programming language C++, we plot the graph of the approximation of the FIF in the non-affine case using the two algorithms.



(a) Probabilistic interpolation scheme, non-affine case with 100 000 points



(b) Deterministic interpolation scheme, non-affine case with 100 000 000 points

Figure 6.1: Approximation of the FIF

After 100 000 steps, in the probabilistic case, using Algorithm 1, we obtain the outcome depicted in Figure 6.1(a). The processing time for generating the data points is 1.039 seconds and the plotting time is 1.3 seconds.

By employing the deterministic Algorithm 2, with the specified parameters of $k = 100$, $n = 100$ and $p = 3$, we obtain the graph displayed in Figure 6.1(b). The combined processing time for both computation and plotting in this particular case is 1474.477 seconds.

6.2.3 Conclusions

The two algorithms considered, the probabilistic one and deterministic one, yield similar outcomes when used to approximate the FIF in both the affine and non-affine scenarios. In the case of the probabilistic Algorithm 1, considerable results are achieved after surpassing 10 000 steps, and the time required to generate the graph visualization is less than 3 seconds.

7. Conclusions

In this thesis, we bring new contributions to fractal interpolation theory. We have demonstrated the existence of FIF via CIFSs composed of Rakotch contractions. In addition, we have introduced an innovative fractal interpolation scheme applicable to possibly sizeable set of data (including uncountable sets of data), proving the existence of a FIF whose graph is the attractor of a PIIFS. Furthermore, based on the theory of fractal interpolation functions we have introduced a new type of IFS, called interpolation type iterated function systems. For this novel I-tIFS we have proven that it has attractor and admits canonical projection. Additionally, we presented two applications of FIF, focusing on epidemic curves (specifically Covid-19) and computational methods for obtaining approximations of the graph of FIFs.

As regards further development, the present thesis opens certain new directions of research. One direction of generalization and new research can be to consider IFSs composed of more general contractions. In this respect, concerning the results in Chapter 3, we expect that CIFSs composed of functions that are of a more general contractive type (for example Matkowski contractions), can be used to obtain the FIF.

Regarding applications of FIF, starting from the results from section 1 of Chapter 6, one direction of research is related to enhancing the accuracy of Machine Learning prediction algorithms using fractal interpolation in the preprocessing step.

In conclusion, this thesis significantly advances fractal interpolation theory by introducing new frameworks and novel concepts. The contributions enhance both the theoretical foundation and practical relevance of fractal interpolation.

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